MATHEMATICS

Equation of Motion in Five Space

THERE exist many models for the description of the gravitational and electromagnetic field. Perhaps the most successful formalism developed is still that of Kaluza[1], with notable improvements by Bergmann[2] since it reproduces exactly the classic equations of the electromagnetic field with gravity. However, there still remained the well-known difficulties associated with the classic treatment of Nature so that generalizations which lead to substantially different equations were made in the hope that there could be included within the new frameworks an adequate theory encompassing quantum-phenomena as well. These hopes have not been realized; but nevertheless it is most difficult to relinquish a theory which by and large has been subject to confirmation—quite strikingly in recent times using techniques based on the Mössbauer effect[3]. In addition to numerous verifications of the original version of the theory, Einstein, Infeld and Hoffmann[4] have shown that there exists a most æsthetic feature, namely, that the field equations themselves determine the classic equations of motion of 'singularities' or 'particles' in the field. Thus it is not necessary to supplement the field equations with ponderomotive equations.

It seems to be of some interest to investigate the equations of motion in a five-dimensional setting to examine departures from the usual results. I will assume that the additional dimension is space-like and take for the field equations:

$$R_{\mu\nu} - g_{\mu\nu}R/2 = 0 \qquad \mu,\nu = 0, 1, 2, 3, 4 \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ the Riemann-Christoffel curvature tensor and R its contraction. '0' indices will be considered temporal, the others spatial with the '4' index reserved for the additional dimension. Now if we seek approximate solutions of the field equations corresponding as closely as possible to the four-dimensional case it is found on reckoning differentiation with respect to X_4 as leading to quantities of the same order of smallness as differentiation with respect to $X_0(= ct)$, and then proceeding in much the same fashion as in reference 2 and 4 'co-ordinates' of the 'particles' will involve X_4 as well as X_0, which would imply that the equations of motion for the position of the 'particles' will be partial differential equations rather than total differential equations. In addition to a field, which, in view of the Kaluza theory, may be interpreted as that arising from a charged 'particle,' there arises another which I make no attempt to interpret here.

For purposes of comparison let us record the results of the calculations for the metric tensors to that order essential to obtain the Newtonian approximation These are:

$$\begin{aligned}
g_{00} &= 1 - c^{-2} \sum_j((12\varkappa M_j + 6\varkappa P_j)/13)V_j \\
g_{0s} &= (6\varkappa/13)c^{-3} \sum_j(3M_j\partial\xi_j^s/\partial t - \\
&\qquad (13/2\varkappa)^{1/2}(cQ_j)\partial\xi_j^s/\partial X^4)V_j \\
g_{4s} &= (6\varkappa/13)c^{-3} \sum_j((13/2\varkappa)^{1/2}Q_j\partial\xi_j^s/\partial t - \\
&\qquad 3P_jc\partial\xi_j^s/\partial X^4)V_j \qquad (2) \\
g_{40} &= -6c^{-2}(\varkappa/26)^{1/2} \sum_j Q_jV_j \\
s &= 1, 2, 3
\end{aligned}$$

where

$$V_j = ((X^s-\xi_j^s)(X^s-\xi_j^s))^{-1/2} \tag{3}$$

ξ_j^s, the sth co-ordinate of the jth 'particle,' and M_j and Q_j its mass (gm.) and charge (E.S.U.) respectively. P_j may be considered another parameter describing another property of the jth particle. It is to be noted that $\varkappa$ is Newton's constant of gravity and that the dimensions of the P_j is that of $(\varkappa)^{-1/2}$ times charge (E.S.U.). M_j, Q_j and P_j could be functions of $X_0(= ct)$ and X_4 satisfying:

$$\begin{aligned}
\partial M_j/\partial t - (13/18\varkappa)^{1/2}c\partial Q_j/\partial X^4 &= 0 \\
\partial Q_j/\partial t - (18\varkappa/13)^{1/2}c\partial P_j/\partial X^4 &= 0
\end{aligned} \tag{4}$$

while the partial differential equations of motion for the jth particle in the Newtonian approximation turn out to be:

$$\begin{aligned}
&\partial^2\xi_j^s/\partial t^2 - (1/3)(cQ_j/M_j)(26/\varkappa)^{1/2}\partial^2\xi_j^s/\partial X^4\partial t \\
&\qquad + (P_jc^2/M_j)\partial^2\xi_j^s/\partial X^{42} \qquad (5) \\
&- \sum_{i\neq j} (\varkappa M_i + Q_iQ_j/M_j - 4\varkappa P_jM_i/(13M_j) \\
&\qquad + \varkappa P_iP_j/M_j - 4\varkappa P_i/13)\partial V(i,j)/\partial\xi_j^s = 0
\end{aligned}$$

where

$$V(i,j) = ((\xi_i^s - \xi_j^s)(\xi_i^s - \xi_j^s))^{-1/2} \tag{6}$$

with the time t in seconds units and X^4 the additional co-ordinates expressed in cm. units.

It is to be noted that even for an isolated 'particle' or systems of 'particles' far removed from one another the co-ordinates ξ_j^s satisfy the second-order equation obtained by considering only the first three terms of equation 5 which may be expressed as:

$$(\partial/\partial t - c_{j+}\partial/\partial X^4)(\partial/\partial t - c_{j-}\partial/\partial X^4)\,\xi_j^s = 0 \tag{7}$$

if M_j, Q_j, and P_j are presumed constant. In equation 7 c_{j+} is given by:

$$\begin{aligned}
c_{j\pm} = (Q_j/M_j)(18\varkappa/13)^{-1/2} &\pm \\
((Q_j/M_j)^2(18\varkappa/13)^{-1} &- P_j/M_j)^{1/2})c
\end{aligned} \tag{8}$$

which implies with the f_j^s and g_j^s arbitrary functions of the indicated arguments that:

$$\xi_j^s = f_j^s(c_{j+}t + X^4) + g_j^s(c_{j-}t + X^4) \tag{9}$$

a wave motion for the ξ_j^s which is propagated with velocities $- c_{j+}$ in our $t - X^4$ space.

The insistence that equation 7 be a hyperbolic partial differential equation has as consequence $P_j/M_j < 0$ to assure this property even when $Q_j = 0$. For an electron the velocity of propagation attains a value of the order 10^{21} times the velocity of light. One would be forced to conclude that if it were possible to communicate in such a space it would be a relatively simple task to assess the state of the far reaches of the universe within a considerably shorter interval of time than heretofore and so indeed to consider occurrences almost anywhere to be of a contemporary nature. In any event the structure of the equations of motion which have emerged even for 'free particles' may perhaps provide us with other conceptual possibilities the starting point of which may lead to a theory with less startling consequences.

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