A CONTINUUM THEORY OF RHEOLOGICAL PHENOMENA

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WHEN materials of different kinds are subjected to mechanical actions we may observe a great variety of rheological phenomena, and we shall confine our attention here to a certain group of them which allows a theoretical and experimental investigation on a macroscopic scale. Here then, as in the classical theories of elasticity and hydrodynamics, we can treat the materials as continuous media, and the forces and displacements as vectors which vary continuously in space and time.

Let us consider first what happens in an appropriately chosen cycle of rheological states. We start with a material in a *ground-state*, that is, in a state in which every portion is at rest and free from external forces. We then impose some mechanical action and thereby bring the material into the particular *energized* state the phenomena of which we want to investigate. Finally, to complete the cycle, we suddenly release all external forces and allow the material to drift back into some ground-state (which may or may not be the one from which it was started). We can now define as a characteristic parameter of the energized state considered, and of the phenomena produced therein, the *recoverable strain*, that is, the amount of strain recovered after the release of the forces.

An experimental investigation was carried out under Dr. C. H. Lander[1], and in this we found it convenient to classify a material as *stable* if it regains always the same mechanical properties after completing a cycle from one ground-state to another, and to refer to it as *general* or *special* in some range of rheological states, according as the recoverable strain reaches in that range finite values, or remains infinitesimally small or zero throughout. The investigation covered a wide range of stable materials comprising such solids and liquids as are homogeneous and isotropic in all their ground-states. It revealed the differences in behaviour between general and special materials rather strikingly under certain conditions, as is illustrated in Fig. 1 for liquids, that is, for materials which flow under any shearing stress, however small.

The experimental arrangement is such that the liquid is sheared in a gap between an outer vessel rotated with an angular velocity that is kept constant at various levels, and an inner member which is held against rotation and either rigidly fixed in position or free to move up and down the axis of rotation. The conditions are so chosen that under the combined actions of the shear imposed at the boundaries, and the forces of gravity and inertia (centrifugal forces), the liquid executes a stationary laminar shearing movement.

An analysis of the experimental results shows that in such a motion the stress has its strength so distributed over the various directions in space as to comprise, in addition to the shear stress components, a pull along the lines of flow. We obtained direct experimental evidence for the presence or absence of such a pull by suddenly stopping the motion, and then making cuts perpendicular to the lines of flow. The general liquid used had a time of relaxation sufficiently long for the stress to persist appreciably after the stoppage, and the pull component present in this stress caused the cuts to open up. This pull

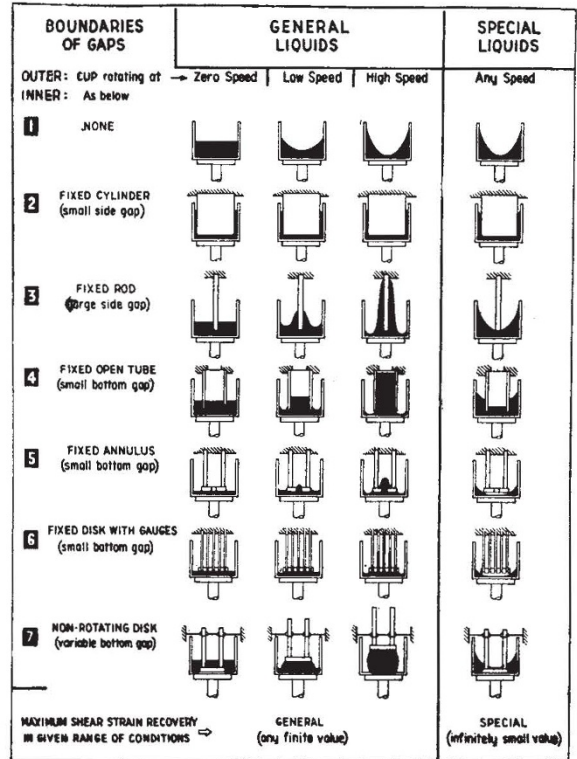| BOUNDARIES OF GAPS	GENERAL LIQUIDS			SPECIAL LIQUIDS
OUTER: CUP rotating at → INNER: As below	Zero Speed	Low Speed	High Speed	Any Speed
1 NONE				
2 FIXED CYLINDER (small side gap)				
3 FIXED ROD (large side gap)				
4 FIXED OPEN TUBE (small bottom gap)				
5 FIXED ANNULUS (small bottom gap)				
6 FIXED DISK WITH GAUGES (small bottom gap)				
7 NON-ROTATING DISK (variable bottom gap)				
MAXIMUM SHEAR STRAIN RECOVERY IN GIVEN RANGE OF CONDITIONS ⇒	GENERAL (any finite value)			SPECIAL (infinitely small value)

Fig. 1. FLOW OF LIQUIDS UNDER ACTIONS OF STATIONARY SHEAR IN GAPS

distinguishes the general liquids from the special ones, being of finite amount or infinitesimally small when the recoverable strain is finite or infinitesimal. (Liquids which are in a certain range 'general' must in the same range be 'non-Newtonian' or 'anomalous'; while liquids which are 'special' may be 'Newtonian' or 'normal', but need not be so.) If, as in our experiments, the lines of flow are closed circles, the pull along these lines strangulates the liquid and forces it inwards against the centrifugal forces and upwards against the forces of gravity. The presence and absence of such a strangulation is illustrated in Fig. 1 under the column divisions headed respectively "General Liquids" and "Special Liquids". I am greatly indebted to Dr. Lander and to his collaborators, Dr. P. Rosin, Dr. H. Fehling, Mr. M. Deutsch and Mr. R. J. Russell[2], for the generous help they gave me in these experiments, and to Mr. S. M. Freeman for many improvements in the experimental technique. My thanks are also due to Dr. F. H. Garner and Dr. A. H. Nissan who, on being informed of the results obtained and while continuing their own lines of research[3], suggested a useful arrangement of pressure gauges (see Fig. 1, row 6).

A continuum theory was developed for the rheological states of general and special materials, along the lines of the classical treatment of elastic continua. As a first approximation a theory was proposed[4], containing an arbitrarily assumed law of elasticity, and from this theory predictions were made for a pull along the lines of flow and for a great variety of rheological phenomena, including such as are reproduced in Fig. 1. The experimental investigation under Dr. Lander (1943–45) then revealed that in a variety of materials, including saponified oils, solutions of rubber, starch, cellulose acetate, etc., the predicted phenomena actually

occur, but with a systematic discrepancy between the strength of the stress components anticipated in the various directions of space and that actually found in quantitative measurements. The discrepancy was rectified, and full agreement reached with all experimental results obtained so far, in a revised theory, in which the arbitrarily assumed law of elasticity was replaced by an empirical law of the form

$$\frac{P_{au} - P_{bv}}{R_{au} - R_{bv}} = \frac{P_{bv} - P_{cw}}{R_{bv} - R_{cw}} =$$
$$\frac{P_{cw} - P_{au}}{R_{cw} - R_{au}} = G(P, R) \quad (1)$$

Here—for the energized state considered—we denote by P_{kl} the kl component of stress, by R_{mn} the mn component of the recoverable strain, measured on the classical scale as the elongation ratio squared minus unity, by a, b, c and u, v, w the numbers 1, 2, 3 or any cyclical permutation of these numbers, and by $G(P, R)$ a function of the three fundamental scalar invariants of stress and strain, and other scalar parameters such as temperature, time, etc. The function $G(P, R)$ may be regarded as a generalized modulus of elasticity, the value of which may vary with the material and the rheological state, the temperature, time, etc. This is best seen when the empirical law (1) is specialized for the case of a stationary laminar shearing movement, where the suffixes 1, 2 and 3 correspond respectively to the directions of the lines of flow, the perpendicular to the shearing plane, and the normal to both these directions, and where the recoverable strain has components R_{mn} which are all zero, except R_{11}, R_{12}, and R_{21}, with $R_{11} = (R_{12})^2 = (R_{21})^2$.

Recently, further progress was made[5] when we succeeded in replacing the empirical law of elasticity (1) by a simple theoretical assumption. We consider an elastic continuum in any energized state, and imagine it cut into two parts by a plane drawn in any direction through some arbitrarily selected point. For every such plane the traction force will define a triad of mutually perpendicular directions, according to its normal, tangential and cross-components; and a second triad is defined by taking similar components of the displacement of the two parts relative to one another when the continuum recovers to a ground-state. We can formally deduce equation (1) from the assumption of a coincidence of these two triads of directions for every plane. This assumption, and the elasticity law derived from it, have been confirmed empirically over wide ranges of materials and mechanical actions; but further experimental investigations are required, and are in fact under way, to ascertain the exact limitations of validity, since a theoretical proof can be given only under certain conditions. Keeping within the range of elastic continua which are isotropic in all the ground-states, we can prove the validity of the assumption (a) in all classes of rheological states for certain orientations of planes, and (b) for all possible orientations of planes in certain classes of rheological states, including all classes restricted either to infinitesimal small strains, or to any stresses and strains with cylindrical symmetry.

[1] Lander, C. H., Melchett Lecture 1945, Institute of Fuel.
[2] Russell, R. J., Ph.D. Thesis, Imperial College, London, 1945.
[3] Garner, F. H., and Nissan, A. H., *Nature*, **158**, 634 (1946).
[4] Weissenberg, K., *Arch. Sci. Phys. et Nat.*, v, **17**, 1 (1934).
[5] Weissenberg, K., Conference of the British Rheologists' Club, London, 1946.

EMERGENCY SCIENTIFIC RESEARCH BUREAU, EIRE

THE report of the Emergency Scientific Research Bureau, Eire, covering the period February 21, 1941–March 31, 1945*, which was set up with Prof. J. J. Dowling as chairman, gives brief details of the work of the Bureau.

A large part of the Bureau's experimental work dealt with fuel problems, and the most important of these included the manufacture of charcoal from turf, the efficiency of mobile producer plants, and the production of town gas under emergency conditions. Work on the production of turf charcoal led to the design, erection and operation by the Turf Development Board, Ltd., of a plant on the Turraun Bog, from which charcoal suitable for use as a producer fuel was obtained with an output of up to 22 tons a week. Work on town gas resulted in a study of the possibilities of using mixtures of turf and coal, and in modifications in plant which to some extent eased the difficulties being experienced at one of the large gas works. One of the first problems studied was that of mobile gas producers, including the use of an electric filter for the gas; but although such a filter can be constructed, it is not much more efficient, is more expensive and requires much more careful servicing. The drying of turf and the development of a gas specific gravity meter were other investigations in the same field.

Much attention was given to chemicals for the manufacture of matches, including the manufacture both of phosphorus and of potassium chlorate; while a process was also worked out for the manufacture of sulphur by dry distillation from spent oxide. Plant was also developed for the supply of compressed carbon dioxide and ammonia for refrigeration purposes, for aluminium sulphate for water purification and paper manufacture, and for the manufacture of formalin by oxidation of methyl alcohol.

A geophysical survey of mineral deposits of economic importance, and the increase in output to 20 kW. of the existing 1·5 kW. short-wave Marconi transmitter at Moydrum were other tasks undertaken.

The maintenance of such essential medical supplies as anæsthetic ether, glycerol, insulin, and vitamins, and of supplies of yeast and flour for breadmaking, and the manufacture of citric acid by fermentation were other subjects investigated.

Experimental work was also carried out on the production of carbon black by combustion of used lubricating oil, the spectrographic analysis of steel, production of ferro-silicon, electro-deposition of ductile pure iron, carburizing steel for knife-blades, refractory bricks, carbon electrodes, replacements for storage batteries, the recovery of zinc from used primary cells and the alkali treatment of Clare phosphate to give a satisfactory fertilizer. A process was developed for a photo-engraving glue from fish-skins. Work was done on acetone, and processes for ethyl acetate from 80 per cent acetic acid, for agar-agar from seaweed, for an insecticide by extraction of offal tobacco, and a substitute for cutch as a preservative for fishing nets, were also worked out by the Bureau. A modified method for the determination of fat in milk was devised. Advice was given on the use of substitutes for imported wool oil, and for economy and efficiency in the use of lubricating oils.