and hence Milne's predicted value of ρ is of the order of 10^{-27} gm./cm.³. On the other hand, in the Einstein universe, which is the basis of Eddington's system, space is hyperspherical; consequently,

$$M = 2\pi^2 \rho R^3. \quad (8)$$

With the value assigned by Eddington to the Einstein radius of space, ρ is again of the order of 10^{-27} gm./cm.³, but at the present epoch of expansion Eddington considered that the radius is about five times the Einstein radius, giving ρ of the order of 10^{-29} gm./cm.³.

Hence, rival theories assign numerically different values for the local density distribution. Nevertheless, they agree in isolating mass parameters of the same order of magnitude. As a tentative physical explanation of this theoretical phenomenon, it is suggested that 10^{56} gm. is the order of mass which a homogeneous and spherically symmetrical continuous material system must possess if: (i) its gravitational potential energy is comparable with its inertial potential energy; and (ii) its radius is of the order of 2×10^9 light-years.

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¹ Milne, *Proc. Roy. Soc., A*, **154**, 43 (1936).

² Milne, *Quart. J. Math.*, **5**, 64 (1934).

³ Milne and McCrea, *Quart. J. Math.*, **5**, 73 (1934).

⁴ Heckmann, "Theorien der Kosmologie" (1942).

⁵ Milne, *Proc. Roy. Soc., A*, **154**, 43 (1936).

⁶ Hubble, "Observational Approach to Cosmology" (1937), 20.

⁷ Eddington, *Mon. Not. Roy. Ast. Soc.*, **104**, 203 (1944).

Cascade Showers under Thin Layers of Materials

In a previous paper, Bhabha and Chakrabarty¹ discussed the general theory of the production of cascade showers. The solution obtained there was finally evaluated by the saddle-point method, and it was shown that the same expression cannot be used for small values of t and y_0 , where t is the thickness of the material in radiation units and y_0 is $\log(E_0/\beta)$, where E_0 is the energy of the primary particle and β is the critical energy for the material. Consequently in that paper numerical values were obtained only for values of $y_0 \geq 3$ and $t \geq 2$.

It can be seen, however, that $P(E, t)dE$, which represents the number of particles (electrons and positrons) having energies between E and $E + dE$ (produced by a primary particle of energy E_0) at a depth t below the surface of the material, is exactly given by

$$P(E, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{rt} dr \cdot \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E}\right)^s \left\{ \sum_{n=0}^{\infty} \left(-\frac{\beta}{E_0}\right)^n \cdot \frac{\Gamma(s)}{\Gamma(s-n)} \cdot \frac{1}{\pi} \int_0^{\pi} \frac{D+r}{(r+\lambda_s-i)(r+\mu_s-i)} ds \right\} ds, \quad (1)$$

where λ_s, μ_s are functions of s (see ref. 1). For convenience in calculations, we can express (1) in the form

$$P(E, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{rt} dr \cdot \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E + \beta g(s, r)}\right)^s f(s, r, \beta) ds, \quad (2)$$

where

$$f(s, r, \beta) = f(s, r) - \frac{\beta}{E_0} f_1(s, r) + \left(\frac{\beta}{E_0}\right)^2 f_2(s, r) - \left(\frac{\beta}{E_0}\right)^3 f_3(s, r) + \dots \quad (3)$$

and $g(s, r)$ is to be so chosen as to make the series (3) rapidly convergent. It can be shown that if we take

$$f_0(s, r) = \frac{D+r}{(r+\lambda_s)(r+\mu_s)}, \text{ and } g(s, r) = f_0(s+1, r),$$

then by taking only the first term $f_0(s, r)$ in place of $f(s, r, \beta)$ in (2), we get practically the whole of $P(E, t)$; and this is particularly so for smaller values of t . We therefore have, to a very high degree of accuracy,

$$P(E, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{rt} dr \cdot \frac{1}{2\pi i E_0} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(\frac{E_0}{E + \beta g(s, r)}\right)^s \cdot \frac{D+r}{(r+\lambda_s)(r+\mu_s)} ds, \quad (4)$$

The integral in (4) can be evaluated by the saddle-point method for all values of t and y_0 , and as such is an improvement on the solution given in ref. 1. Hence if $\exp \psi(s, r)$ represents the integrand of (4), then

$$E_0 P(E, t) = \frac{1}{2\pi} \left[\exp \psi(s_0, r_0) \cdot \left\{ \frac{\partial^2 \psi}{\partial r^2} \cdot \frac{\partial^2 \psi}{\partial s^2} - \left(\frac{\partial^2 \psi}{\partial r \partial s} \right)^2 \right\}^{-1/2} \right]_{r=r_0}^{s=s_0} \quad (5)$$

where s_0 and r_0 satisfy the equations $\frac{\partial \psi}{\partial r} = 0 = \frac{\partial \psi}{\partial s}$, simultaneously.

The total number of particles $N(t)$, at any depth t in the shower-producing layer, can be easily obtained from (4). The values of $N(t)$ thus obtained for some different values of y_0 and t are given in the accompanying table.

VALUES OF $N(t)$ FOR A PARTICLE EXCITED SHOWER

$t \backslash y_0$	1	2	3	4	5	6
0.2	1.12	1.20	1.27	1.34	1.40	1.48
0.5	1.15	1.50	1.84	2.18	2.51	2.85

Janossy and Tzu² have suggested that the solution given by Bhabha and Chakrabarty¹ can also be used for smaller values of t , provided we retain also the term containing $\exp(-\mu_s t)$ occurring there in the integrand. That this is not so will be evident when the values of $N(t)$ for $t = 0.5$ and 0.2 are obtained from that expression. For smaller values of E_0 which are interesting from the practical point of view, the error comes to about 40 per cent for $t = 0.5$ and to 100 per cent or more in the case of $t = 0.2$.

With the above solution it is now possible to deduce the energy spectra of the shower particles at all thicknesses and for all energies of the primary particle.

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¹ Bhabha and Chakrabarty, *Proc. Roy. Soc., A*, **181**, 267 (1943).

² Janossy and Tzu, *Nature*, **157**, 624 (1946).

Turbulent Flow in Alluvium

SEVERAL years ago, I put forward¹ two original equations for the flow of water in alluvium: the first for the wetted perimeter P in terms of the discharge Q , the second for the mean velocity V in terms of the hydraulic mean depth R and water surface slope S . Both expressions had the curious characteristic that they appeared to be independent of the grade of alluvium transported. They may be written in general terms

$$P = 'x' Q^{1/2} \quad (1)$$

$$V = 'y' R^{2/3} S^{1/3} \quad (2)$$

The equations were empirical and derived from a mass of hydraulic data. Their simplicity gave cause for the hope that eventually a dynamic theory to justify them would be evolved, and basic dimensionally homogeneous equations for the turbulent flow of water in alluvium derived. In this connexion the following analysis is fruitful.

From equation (1)

$$(RV)^2 = Q/x^2.$$

From equation (2)

$$V^5 = y^3 (RV)^2 S.$$

Eliminating (RV) , which is a proportional to the Reynolds number,

$$S = x^2 V^5 / y^3 Q; \quad (3)$$

or, on introduction of ' g ' the acceleration due to gravity,

$$S = KV^5 / g^2 Q, \quad (3a)$$

in which the coefficient is given by the equation

$$K = g^2 x^2 / y^3 \quad (4)$$

Equation (3a) is dimensionally homogeneous, and the coefficient K must be either a pure number or a pure number multiplied by one or more dimensionless numbers to a very low power, and of insufficient weight to affect the approximate constancy of the coefficient K . The computed value of K is clearly of great moment.

In 1930 I assigned a value of 2.67 to ' x ' and of 16.1 to ' y '. In 1937 the Punjab Irrigation Research Institute² confirmed my equation (1) from a more extended range of observations and assigned to ' x ' the value of 2.80. Other observations on other collections of data have given values of ' x ' ranging from 2.5 to 3.0. The Punjab collection is more authoritative. My latest value of ' y ' is 16.0. It would not appear possible from observations on canals to obtain values of these coefficients with an error of less than 5 per cent. If for ' x ' a value of 2.81 is taken, and for ' y ' a value of 16.0, ' g ' being given a value of 32.2, the value of K as computed from equation (4) is precisely the round number 2. The question immediately arises whether this is a pure number. The following theorem which I have derived enables us to arrive at a tentative conclusion on this point.

If we consider the horizontal and vertical axes of an ideal asymmetrical alluvial channel cross-section, W can be taken as a length characterizing the effective width of the channel, and D a length characterizing the effective depth. The intersection of the axes at the centre of the water surface may be regarded as the polar origin of generation of the channel section by the turbulent water flowing with a discharge of Q . There are thus two lengths involved, $\frac{1}{2}W$ and D .