

### Elimination of Divergencies in Quantum Electrodynamics and in Meson Theory

A DIVERGENCE-FREE theory of the classical electron has been given by Wentzel, and then, with other methods, by Dirac. In quantum theory these methods give an elimination of part of the infinite expressions. Making further the assumption of positive- and negative-energy photons, Dirac eliminates all the infinite expressions.

The theory of the radiating classical electron is governed by the wave equation. Hadamard gave by his concept of 'partie finie' a meaning to divergent integrals which appear in the formal solution of the wave equation. M. Riesz<sup>1</sup> gave a new mathematical treatment of the wave equation, showing that the divergent integrals can be avoided. This method of solution applied to the electron gives an equation of motion which is identical with Dirac's equation for the classical electron. Applications of the method to the classical theories of the electromagnetic and meson fields are described in a paper by Fremberg<sup>2</sup>.

We will now use these mathematical methods in quantum theory, treating as examples expressions, for example, the interaction energy, where divergencies appear already in second approximation when using ordinary methods. We use the formalism in which the dynamical equations are operator equations, giving the variation with  $t$  of the matrices. We treat first the interaction of a nucleon field  $\psi$ , which satisfies Dirac's wave equation, and a scalar meson field  $\Phi$ .

$$\left[ \frac{i\hbar}{c} \frac{\partial}{\partial t} + i\hbar\alpha_i \frac{\partial}{\partial x_i} - \beta mc \right] \psi = \frac{g}{c} \Phi \beta \psi$$

$$\square \Phi + \kappa^2 \Phi = -g\psi^* \beta \psi.$$

We assume the usual commutation relations. Using second quantization for the nucleon matrix wave function  $\psi(x,t)$ , we expand  $\psi$  in a series of plane waves with matrix coefficients  $a_n(t)$ . We discuss the self-energy terms in second approximation, first the term

$$W_1 = \frac{g^2}{2} \iiint dV \psi^* \beta \psi \Phi^1.$$

Here  $\Phi^1$  satisfies the meson equation with the right-hand member  $-\psi^* \beta \psi$ . The solution of this equation by analytical continuation according to Riesz is

$$\Phi^1: \alpha = - \iiint_{D_S^P} \psi^*(Q) \beta \psi(Q) \left( \frac{rPQ}{\kappa} \right)^{\frac{\alpha-4}{2}} \frac{J_{\frac{\alpha-4}{2}}(\kappa rPQ)}{2\pi 2^{\alpha/2} \Gamma\left(\frac{\alpha}{2}\right)} dQ.$$

Here  $rPQ$  is the Lorentz-distance between the four-dimensional points  $P$  and  $Q$ . The integration over  $dQ = (cdt dx_1 dx_2 dx_3)Q$  is taken over the whole four-dimensional domain bounded by the retrograde light-cone with its top at  $P$  and by the space  $S$  at the time  $t = T$  (in the meson case  $T = -\infty$ ).

$\Phi^1: \alpha$  is an analytical function of  $\alpha$ . The solution of the meson equation is obtained by analytical continuation in  $\alpha$  to  $\alpha = 2$  (see refs. 1 and 2). We insert  $\Phi^1: \alpha$  into  $W_1$  and get an expression  $W_1: \alpha$ . Neglecting the retardation and retaining the terms not zero in the one-nucleon case, we find in the integrand of  $W_1: \alpha$  a sum

$$\sum_n \exp\left(\frac{i}{\hbar} k_n (X_P - X_Q)\right) = \delta(X_P - X_Q).$$

We find:

$$W_1: \alpha = - \frac{g^2 \kappa^3 - \alpha}{4\pi 2^{\alpha/2} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \tau^{\frac{\alpha-4}{2}} J_{\frac{\alpha-4}{2}}(\tau) d\tau.$$

The integral is convergent for  $3 < \alpha < 5$  and is for these  $\alpha$ 's equal to

$$2 \frac{\alpha-4}{2} \Gamma\left(\frac{\alpha}{2} - \frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right).$$

This expression, and therefore  $W_1: \alpha$ , can be continued analytically to all  $\alpha$ . Now  $\Phi^1$  has to be a solution of the meson wave equation. This is achieved by putting  $\alpha = 2$ . So we find that  $W_1$  or the energy (when neglecting the retardation) of the meson field created by a point nucleon has a finite value  $g^2 \kappa / 8\pi$ .

We make a comparison with the electrostatic energy of a point charge. The equations for this case are obtained by substituting in the interaction terms 1 and  $-1$  for the first and second  $\beta$ , respectively, which only alters the sign of the energy, and then by putting  $\kappa = 0$ . Thus we find the value zero for the electrostatic energy of a point charge. We have also found this value in another way<sup>3</sup>.

We proceed to the other self-energy term of second approximation, the integrand of which is  $(\psi^* \beta \psi \psi + \psi^* \beta \psi^2) \Phi^0$ . It is the energy of the forced oscillations of the nucleon produced by the meson vacuum field fluctuations. When assuming Dirac's theory of positive- and negative-energy mesons, the divergence diminishes by one order. A fuller account of the meson case will be given elsewhere<sup>4</sup>.

We wish to stress the circumstance that in the calculations above, the solution of the meson field equation has been obtained by analytical continuation, whereas the Dirac wave equation has been solved in the ordinary way. The Dirac equation is also, however, a hyperbolic

equation, the solution of which and, thereby, the corresponding energy expressions, can also be given by analytical continuation. I hope to be able to return to these questions later.

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<sup>1</sup> Riesz, M., Congrès internat. d. math., Oslo, 1936, II, 44. Coafér. à la Réunion internat. d. Math. à Paris, 1937 (1939).

<sup>2</sup> Fremberg, N. E., Proc. Roy. Soc., in the press.

<sup>3</sup> Gustafsson, T., Lunds Fysiogr. S. Förh., 15, 28 (1945).

<sup>4</sup> Gustafsson, T., Lunds Fysiogr. S. Förh., 16, 2, in the press.

### Preparation of Synthetic Quartz

A RECENT communication directs attention to the crystallization of silica glass to give small quartz crystals<sup>1</sup>. My experience in the hydrothermal synthesis of quartz may be of interest in view of this letter and of the industrial importance of this mineral.

A partial or complete crystallization of silicic acid gel was obtained by heating the gel in an autoclave with potassium carbonate or sodium carbonate solution for three days at  $\sim 350^\circ\text{C}$ . to  $390^\circ\text{C}$ . Quartz was the only crystalline species present. The birefringent crystals were small but often beautifully formed, and frequently revealing the striations and hemihedral faces seen in perfect natural crystals. Along with other species such as analcite, quartz was also obtained by hydrothermal crystallization of aluminosilicate gels. These were prepared by stirring sodium aluminate solution into silicic acid suspension, and evaporating to dryness; the crystallization was effected in media of water, sodium carbonate solution and sodium bicarbonate solution at temperatures ranging between  $\sim 255^\circ\text{C}$ . and  $\sim 355^\circ\text{C}$ . After crystallization, the pH of the cold mother liquor in all cases lay in the range 8.5-10.5, that is, the liquor was alkaline. Gels of compositions  $\text{NaAlO}_2 \cdot 6 \cdot 15\text{SiO}_2$ ;  $\text{NaAlO}_2 \cdot 5 \cdot 47\text{SiO}_2$ ;  $\text{NaAlO}_2 \cdot 4 \cdot 68\text{SiO}_2$ ;  $\text{NaAlO}_2 \cdot 4 \cdot 10\text{SiO}_2$  usually gave *inter alia* yields of quartz, though often small in amount compared with other species. Mixtures poorer in silica than the above did not give any quartz under these alkaline conditions. It was concluded that alkali is a good selective mineralizer for quartz in silica-rich mixtures only, from which under the above conditions quartz is easily formed.

It must not be thought that synthetic quartz formation is rarely reported. Experiments have been made quite similar to those described by Wooster and Wooster<sup>1</sup>, in which various glasses (crown, borosilicate and flint<sup>2</sup>; thuringian and obsidian<sup>3</sup>; silica glass<sup>4</sup>; glass powder<sup>5</sup> and minerals (muscovite<sup>6</sup>; tridymite and cristobalite<sup>4</sup>) have been crystallized or recrystallized. Crystallizations of various gels and mixtures (for example, silica gel<sup>7</sup> and sol<sup>8,9</sup>; alkali,  $\text{Al}(\text{OH})_3$  and  $\text{Si}(\text{OH})_4$ <sup>10</sup>;  $\text{K}_2\text{SiO}_3$ ,  $\text{KOH}$  and  $\text{Al}(\text{OH})_3$ <sup>11,12,13</sup>; and a variety of others) have been equally successful<sup>14,15</sup>. There is thus abundant material to draw on in attempts to make quartz on a large scale.

It is also of interest that the formation of tridymite<sup>2,3,11,15,16</sup> and cristobalite<sup>17,18</sup> has been reported under similar hydrothermal conditions to those for making quartz, and so in a metastable temperature range. It is thus likely that all the variables are not yet realized.

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<sup>1</sup> Wooster and Wooster, *Nature*, 157, 297 (1946).

<sup>2</sup> Morey and Bowen, *J. Soc. Glass Tech.*, 11, 97 (1927).

<sup>3</sup> Konigsberger and Muller, *Centralbl. Min.*, 339, 353 (1906).

<sup>4</sup> Fenner, *J. Wash. Acad. Sci.*, 2, 471 (1912).

<sup>5</sup> Bruhns, *Neues Jahrb. Min.*, (ii), 62 (1889).

<sup>6</sup> Friedel, *Bull. Soc. min. franc.*, 19, 5 (1896).

<sup>7</sup> de Senarmont, *Ann. chim. phys.*, (3), 32, 142 (1851).

<sup>8</sup> Chrustchoff, *Bull. Soc. min. franc.*, 10, 31 (1887).

<sup>9</sup> Chrustchoff, *C.R. Acad. Sci., Paris*, 104, 602 (1887).

<sup>10</sup> Friedel and Sarasin, *Bull. Soc. min. franc.*, 2, 113 (1879).

<sup>11</sup> Friedel and Sarasin, *Bull. Soc. min. franc.*, 2, 158 (1879).

<sup>12</sup> Friedel and Sarasin, *Bull. Soc. min. franc.*, 3, 25 (1880).

<sup>13</sup> Friedel and Sarasin, *Bull. Soc. min. franc.*, 4, 171 (1881).

<sup>14</sup> Bauer, *Z. phys. Chem.*, 42, 567 (1902).

<sup>15</sup> Allen, *Amer. J. Sci.*, (4), 23, 297 (1906).

<sup>16</sup> Schlaepfer and Niggli, *Z. anorg. Chem.*, 87, 52 (1914).

<sup>17</sup> Weil, *C.R. Acad. Sci., Paris*, 181, 423 (1925).

<sup>18</sup> Ewell and Insley, *J. Res. U.S. Bur. Stand.*, 15, 173 (Reprint 819) (1935).

### Scattering of Polarized Light by a Colloidal Graphite Solution

THE first Lord Rayleigh dealt with the scattering of plane polarized light seventy-five years ago<sup>1</sup>. In the theoretical treatment it is assumed that the colloidal particles are spherical. A newly prepared solution of colloidal sulphur is usually employed to demonstrate his results, and the observations indicate that no light is scattered in the plane containing the electric vector. (This is the direction marked on 'Polaroid' disks, and I will here adopt the convention of calling it the plane of polarization.)

Consider a beam of plane polarized light directed vertically downward on a cylindrical glass absorption cell with its upper surface covered with a plane window. The cell is filled with a weak suspension of 'Aquadag' colloidal graphite in water. Observations are carried out in a horizontal plane, and either the observer moves around the tube in a circle, or the polarizing 'Polaroid' is rotated. The diffused light appears almost equally bright when viewed from any direction horizontally. If, however, a 'Polaroid' is placed before the eye, it is found that the light is nearly completely plane polarized, the inclination of the plane of polarization being related to the angular position of the observer as indicated in the diagram, where  $PP$  repres-