THEORY OF SEA WAVES

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Recent contributions in Nature on waves[1] have been confined chiefly to cases in which the profiles are trochoidal, but they are not trochoidal when the steepness or the ratio of height of wave to depth of water is large, and adjustments are called for when the waves are near breaking point. The present purpose is to consider to what extent theory provides the clue, and what further information is required.

The following abbreviations are used:

L = length	E = energy per sq. ft. of water surface.
H = height	V = wave speed through water
D = depth of water	G = speed of energy
$S = H/L$ = steepness	q = elevation of orbits
C = stream (negative for foul)	

Suffixes:	s = surface	k = kinetic
	w = (speed) through water	p = potential
	g = ,, over ground	o = deep slack water

Units: lb., ft., sec.

To start at the beginning, L, H, D are necessary to specify a wave, and its behaviour will depend on the ratios D/L, H/L, H/D. Theory as to D/L is fairly complete, provided there is no complication due to large value of either of the other ratios; for if the latter are small, the waves will be trochoidal for all values of D/L.

Next, it will be convenient to settle the narrowest practicable limits within which D/L cannot be regarded as either large or small. So far as the equations for wave speed are concerned, it may be taken as large provided it is at least 0·4, and as small if it is not more than 0·04, because then the limiting values, $\sqrt{gL/2\pi}$ and $\sqrt{gD}$, will not differ by more than one per cent from $\sqrt{gL/2\pi} \times \sqrt{\tanh 2\pi D/L}$. But it must be remembered that when D/L is 0·4, G/V, that is, $1-(L/V)(dV/dL)$, will be 6 per cent above its deep-water figure of 0·5, and that the run at the bottom will already be a sixth of the wave height.

It will also be convenient to coin some term for waves having D/L between the limits in question. 'Transitional' is suggested, as waves in gradually shoaling water have to pass through a state that is transitional between deep water and 'long'.

When S or H/D is not small, some factor, say A, has to be inserted in the expression for V. It seems that this factor is only required when S or H/D is so large that the waves are no longer trochoidal. Stokes[2] gives $A = \sqrt{1 + \pi^2 S^2}$, that is, $1 + 5S^2$, for deep water. Rayleigh[3] confirms this, but going to closer approximation makes $A = \sqrt{1 + \pi^2 S^2 + 1\cdot25\pi^4 S^4}$. The extra term makes a little difference near breaking point.

Michell[4] says that with waves which maintain their form, which presumably means that both crest and trough move at the speed of the wave as a whole, so that the profile keeps symmetrical about the crest, S cannot exceed 0·142 in deep water, and that A is then 1·2—apparently a misprint for 1·1. This compares with 1·12 from Rayleigh's formula. Possibly it is merely intended to be a round figure, so that Rayleigh's may be taken as the more exact.

Wilton[5], professing to carry the argument to a greater degree of precision, puts $S_{\max.}$ at 0·132, with V at $2\cdot9\sqrt{L}$, giving $A = 1\cdot28$. This value for A lies so much above Rayleigh's 'A for S' curve that it is difficult to reconcile the two. The curve would have to be distorted for even moderate values of S to make it hit Wilton's point, and in the absence of any formula for the distortion it will be best to assume that Rayleigh's curve holds good throughout.

No one seems to have evolved an expression for A applicable to transitional waves. It would have to include all three ratios, as the following argument will show. A table was given in Nature of November 14, 1942, p. 582, to show how deep-water waves change when they reach shoaling water. This table indicates that when D/L_0 = (say) 0·20, 0·10, 0·01, S/S_0 = 1·04, 1·3, 5·75, and H/H_0 = 0·92, 0·93, 1·43. It follows that H/D is at least 4, 9, 140 × S_0. As D/L_0 has to become as small as 0·01 before D/L is reduced to 0·04, this shows that even waves of almost negligible deep-water steepness must break before they become 'long', while steepish ones will acquire appreciable values of H/D soon after they become transitional. In effect, while steepness is the only disturbing factor in deep water, H/D also operates with transitional waves, and does so alone with 'long' ones.

Airy[6] (confirmed by Stokes[7] and Lamb[8]) has shown that when H/D is not small, $A = 1 + 1\cdot5\,\eta/D$ for 'long' waves, η being vertical height of water surface above calm-water level, and so negative at the trough. It appears from Lamb that the value of A so obtained only applies to that part of the wave at which the vertical distance in question is measured. So, as $\eta = \frac{1}{2}H + q_s$ and $-\frac{1}{2}H + q_s$ for crest and trough, and q_s for 'long' waves is $H^2/8D$, the difference in speed between crest and trough will be $(1\cdot5H/D)\sqrt{gD}$. The wave will take about $L/\sqrt{gD}$ sec. to cover its own length, and while it does so the crest will gain $1\cdot5L \times H/D$ ft. on the trough ahead, or $1\cdot5H/D$ per foot of travel. So, if the slope of the ground be 1 in X, the gain will be $1\cdot5X.H/D$ for each foot of shoaling.

The consequent steepening of the wave front will hasten breaking, and it seems reasonable to suppose that similar, though slighter, distortion takes place with transitional waves, in which case they ought to break in deeper water if the shoaling is gradual. That is precisely the opposite to what Gaillard[9] found. He made two sets of observations, one on the open coast in Florida, and the other on Lake Superior. The former included ground as steep as 1 in 12, while in the latter the slope only ranged from 1 in 90 to 1 in 30. His explanation is that the undertow, strongest where the ground is steep, causes premature breaking. That seems natural, but he seems to indicate that his conclusion also applies as between the flatter slopes on Lake Superior; and undertow only operates within a wave-length of the shore. If it is merely a question of where breaking starts within that distance, well and good; but if he also found that the steeper of two flattish slopes caused breaking in greater depth, the matter may call for further consideration. It is of some practical importance, as it affects the protection that an off-shore shoal may be expected to afford to a breakwater, according to its ability to make the larger waves break before they reach the actual structure.

Incidentally, a splendid photograph of a wave breaking in Lake Superior at least two wave-lengths from the beach appears opposite p. 123 of Gaillard's book. So far as one can judge, the profile approximates to the theoretical one calculated by Stokes[10], in which, for deep water, the back and front should attain slopes of 30° to the horizontal, and intersect at a clean-cut crest.

Another consequence of the steepening of the wave front is the emergence of long-period waves from what seems to be an absolutely flat sea. At suitable places, well up estuaries, like Porlock or Minehead in England, it is not uncommon to see such waves appear almost spontaneously out of a glassy calm, in the form of steep and quickly growing crests, separated by long flat troughs. They only survive for a few wave-lengths before they break, and their period is in the region of 10 sec. That indicates a distant swell of the order of 500 ft., giving its last kick, and doing so in a way which can be quite spectacular.

Gaillard gives sufficient details of 122 of the breakers on Lake Superior for determining all three ratios, and the results show that breaking does almost always start before the waves become 'long' as suggested above. Thus, while D/L ranged from 0·16 to 0·04, H/L from 0·038 to 0·116, H/D from 0·38 to 0·97, the majority of the figures cluster around $D/L = 0{\cdot}10$, $H/L = 0{\cdot}055$, $H/D = 0{\cdot}6$. H ranged from 2 to 11 ft.

H/L also modifies the relation between H and E. Steepness entails some constant, say B, changing the ordinary equation into $E = 8BH^2$. Dunkerley[11] has shown that in deep water $B = 1 - \frac{1}{2}\pi^2 S^2$; while Gaillard[12] gives $1 - \frac{1}{2}\pi^2 S^2 . \coth^2 \frac{2\pi D}{L}$ for transitional waves. Dunkerley's value is the special case when D/L is large. In the other special case, when D/L is small, and the waves are 'long', Gaillard's expression reduces to $1 - \frac{1}{2}\pi^2 . H^2/8D^2$. The assumption, in each instance, is that the waves are trochoidal, in which case $E_k = E_p$. Rayleigh[13] has shown that E_k exceeds E_p when deep-water waves are so steep that his values of A operate, but the expressions he gives are scarcely in a form which admits of easy evaluation. Beyond this, there seem to be no theoretical estimates of B when either S or H/D, or both, are large.

A good example of the effects of A and B is afforded when they are taken into account in considering the consequences of a foul stream in deep water. The equation[14] connecting C and L is $C = L . V_0/L_0 - V$. Allowing for A, and taking $G_w = \frac{1}{2}V$, the limiting stream, which will cause breaking however small S_0 may have been, becomes $-\frac{1}{4}A^2V_0$, instead of $-\frac{1}{4}V_0$; though it still equals $-\frac{1}{2}V$, for V is changed from $\frac{1}{2}V_0$ to $\frac{1}{2}A^2V_0$. Thus, as Rayleigh's A is 1·100 or 1·117, according as to whether Wilton's or Michell's figure for $S_{\text{max.}}$ be accepted, the limiting stream is increased by 20 or 25 per cent. This gives more probable values than those found before, and which were described at the time as apparently on the low side. With Wilton's A as well as his $S_{\text{max.}}$, the increase would be 65 per cent. The other controlling equation, arising from the crowding up of the energy, is $H/H_0 = \sqrt{E/E_0} = \sqrt{G_0/G_g}$. This becomes $H/H_0 = \sqrt{B_0E/BE_0} = \sqrt{B_0G_0/BG_g}$.

There are thus two equations with three variables, L, H, C, so that L and H can be determined for values of C. There seems to be no doubt as to the value of A, except near breaking point, but the probability is that Dunkerley's B is too high for waves too steep to be trochoidal. Luckily, only $\sqrt{B}$ is involved, so it should not make much difference in any event. Results applicable to all values of L_0 are given below, both when Rayleigh's A is taken into account, and, for comparative purposes, also when $A = 1$. B is taken as unity in each instance. It will be seen that A increases the limiting foul stream by nearly a knot for 100-ft. waves, when $S_0 = 0{\cdot}05$ or 0·025. For other values of L_0 the increase would be proportional to $\sqrt{L_0}$, but the percentage loss in length at breaking point entirely depends on S_0.

Rayleigh's A; $B = 1$			$A = B = 1$		
L	S	C	L	S	C
100	0·050	0·0	100	0·050	0·0
90	0·060	—1·2	90	0·060	—1·15
80	0·075	—2·45	80	0·074	—2·1
70	0·098	—3·85	70	0·095	—3·1
			60	0·128	—3·95
61½	0·132	—5·4	59	0·132	—4·0
60	0·142	—5·8	57	0·142	—4·15
100	0·025	0·0	100	0·025	0·0
90	0·030	—1·15	90	0·030	—1·15
80	0·037	—2·25	80	0·037	—2·1
70	0·048	—3·25	70	0·048	—3·1
60	0·065	—4·25	60	0·064	—3·95
50	0·099	—5·5	50	0·092	—4·7
46	0·132	—6·45	42½	0·132	—5·15
45	0·142	—6·7	41	0·142	—5·2

To apply this table to other values of L_0, increase L proportionately to L_0, and C to $\sqrt{L_0}$.

Relevant equations, $C/\sqrt{L_0} = (L/L_0 - A\sqrt{L/L_0}) \times \sqrt{g/2\pi}$

$$S/S_0 = (2L/L_0 - A\sqrt{L/L_0})^{-\frac{1}{2}} \times L_0/L$$

A_0 is ignored, being almost unity.

There is one small point about this table. If $\delta V/\delta L$ and $\delta C/\delta L$ be calculated from the part based on $A = B = 1$, the results will conform with the interdependent equations $G_w = V - L . dV/DL = \frac{1}{2}V$ and $G_g = L . dC/dL$, but they will be quite discordant if calculated from the other part. That does not mean to say that A alters G_w from $\frac{1}{2}V$ when change of stream is met in deep water. A similar apparent discrepancy will be found in Rayleigh's table[15] for waves running into shoal water, for if the figures for $\delta V/\delta L$ obtained from that table be applied to the equation for G, the result will be zero for all depths, and that would be nonsense. Perhaps the explanation is that, in one case, an extra variable A arising from change in S, and so from change in stream, is introduced, while in the other there is the extra variable D; and that the waves, at any one moment, cannot 'foretell' whether they are going to meet a different stream or depth at the next one, and so, for the time being, must behave as if A, or D, were constant. Thus modification of the value of $\delta V/\delta L$, due to future change in stream or depth, cannot affect the value of G, and make it ambiguous, before the change takes place.

Lastly, it seems that waves in mid-ocean do not attain a steepness sufficient to make it likely that A or B will materially affect their rate of growth.

[1] NATURE, **148**, 226 (1941); **149**, 219 and 584 (1942); **150**, 581 (1942).

[2] Stokes, "Math. and Phys. Papers", **1**, 211.

[3] Rayleigh, "Scientific Papers", **6**, 13. (Rayleigh's "4/5" is misprint for "5/4".)

[4] *Phil. Mag.*, (5) **36**, 430 (1893).

[5] *Phil. Mag.*, (6) **23**, 1055 (1913).

[6] "Encyclopedia Metropolitana", "Tides and Waves", art. 208.

[7] Stokes, "Math. and Phys. Papers", **1**, 172.

[8] Lamb, "Hydromechanics", 6th ed., 262.

[9] Gaillard, "Wave Action", 120–123.

[10] Stokes, "Math. and Phys. Papers", **1**, 227.

[11] Dunkerley, "Hydraulics", **2**, 54.

[12] Gaillard, "Wave Action", 45–46.

[13] Rayleigh, "Scientific Papers", **6**, 14.

[14] NATURE, **149**, 219 (1942).

[15] Rayleigh, "Scientific Papers", **6**, 8.