The Mathematics of Experimentation

THE first morning of the British Association meeting at Cambridge saw a discussion in Section A* (Mathematics), of exceptionally wide interest to workers in experimental science. The five speakers, three American and two English, had all in recent years engaged in the study of the combinatorial problems underlying modern types of experimental design, aimed at eliminating errors due to heterogeneity of material, and at founding inferences on valid tests of significance.

On one side, the experimental importance of designs such as the Latin square, now widely adopted, has led mathematicians to a more serious study of old problems of which the previous treatment in the mathematical literature has been discontinuous and desultory; on the other, fresh combinatorial possibilities are being explored, capable of meeting the peculiar experimental difficulties encountered in fields at present ill supplied with effective experimental designs.

Prof. C. C. Craig, of Ann Arbor, Michigan, opened the discussion with an account of tests of significance from which the customary basis of the theory of errors, the normality of residual deviations, is completely eliminated. Such tests, using the ancillary information supplied by the data themselves, have been the subject of several recent mathematical papers. Their logical cogency is unquestionable; but their use is limited by their being certainly laborious, and usually believed to be less sensitive than the simpler tests in common use. Prof. Craig exhibited a theoretical approach to the problem of sensitivity, and showed that this is more nearly comparable than is often supposed with that of the ordinary tests based on the theory of errors.

In discussing the enumeration of the Latin and Græco-Latin squares of side 7, Dr. Horace W. Norton, of the Galton Laboratory, London, gave a most interesting account of the transformation sets into which these can be grouped. As has long been recognized, an aggregate of $7! \times 6!$ (3,628,800) Latin square solutions may be represented by a single standard square, having the letters of the first row and column in standard order. By the process of intramutation, the six letters other than the corner element, A, are permuted, and the rows and columns rearranged to the standard order. Thereby sets up to 720 standard squares may be generated, such sets having an invariant diagonal structure, by which possible identifications can be recognized. Choice among the 49 possible corner elements then gives a transformation set of possibly 35,780 standard squares. Finally, permutation of the three categories, rows, columns and letters, may yield 6 adjugate sets, or 211,680 standard squares in all. In the less degenerate sets, recognition of possible identities among corner elements, and among categories, is easily accomplished by mapping the positions of the 2×2 Latin squares, of which usually, but not always, there are a number intercalated in any square under investigation. These big sets are sufficiently large to be useful in enumerating the entire family of 7×7 Latin squares, which do not perhaps exceed 25 million in number. The essence of the procedure is that we should be able to test expeditiously whether any given square is a member of a set already known, or whether a new set has been discovered.

The reversal of an intercalated 2×2 square will generate a new square, not usually belonging to the original set. Most of the sets so far known (about 100) have been discovered in this way, and it is not impossible that all sets having intercalates form one connected system. The process cannot, however, lead to any set which lacks intercalates altogether, and Dr. Norton made a fruitful suggestion in pointing out that reversal may also be applied to intercalated 2×3 rectangles, which may lead to such sets. The sets lacking intercalates so far discovered are, however, all involved in Græco-Latin squares, in which the permutation of four categories may generate 24 different adjugate sets having, as Latin squares, four different aspects. The five known Græco-Latin sets all involve in some aspect squares orthogonal to the very degenerate group known as cyclic squares, which, as Jacob had shown, are only 120 in number. The number of Græco solutions of these squares, running into hundreds, contrasts strikingly with the rarity of Græco solutions among the ordinary sets of Latin squares. During the year, Dr. Norton has enumerated about 12 million standard squares, and has incidentally brought to light a structural system of relationship, more important perhaps for our general knowledge of Latin and higher squares, than the simple problem of complete enumeration.

The bearing of such combinatorial researches on practical experimentation was well brought out by the two following speakers, Dr. W. J. Youden, of the Boyce-Thompson Institute, New York, and Mr. Frank Yates, of Rothamsted Experimental Station. Dr. Youden discussed problems confronting the plant physiologist or plant pathologist using as experimental material the successive leaves of a number of different experimental plants. He gave data to explain his experience that both the individuality of the plant and the ordinal value of the leaf very largely affect the reaction observed. Precision is greatly increased if both these factors can be simultaneously eliminated, as in the Latin square. The number of substances or dilutions which it is necessary to test simultaneously, is, however, often greater than the number of leaves on each plant. In such cases, the elimination of plant individuality may be effected by Yates's method of incomplete randomized blocks, in which although all treatments cannot be applied to any one plant, every treatment is tested in the aggregate on the same plants with one or more complete replications of the alternative treatments. In other words, every *pair* of treatments occurs equally frequently on the same plant. Youden showed that this advantage may be combined, as in the Latin square, with having one complete replication at each leaf level.

Those solutions of the problem of incomplete blocks, of which tables have now been published, in which the number of replications is equal to the number of elements in a block, furnish the basis for the formation of Youden's squares. In these the rows and columns are equal in number to the treatments, or to the plants, while the letters are only as numerous as the number of replications, or of leaves on each plant. Every row and column must contain all letters once, the remaining spaces being unoccupied. Further, every pair of rows (or columns) must be occupied simultaneously in the same number of columns (or rows). When the incomplete block solution is known, little difficulty has been found in completing the square. The non-existence of an incomplete-block solution is, however, a difficult matter to establish, and Dr. Youden indicated that nearly exhaustive trials so far suggested the non-existence of the (arithmetically) possible 22×22 square having seven letters.

It was gratifying to observe that a large and predominantly mathematical audience showed the greatest interest in Dr. Youden's account of practical experimental requirements, and in the variety of applications which the known solutions open up in biological material.

In the short time available, Mr. Yates, whose work in this field is widely known, confined himself to explaining the logical genesis of the very beautiful Lattice square, commencing from the simple, triple or multiple lattice. This whole group of designs is adapted to the requirements of the plant breeder, who may need to test two hundred or more varieties in a single year. If the number of varieties is a perfect square, they may be cross-classified in a square lattice, of which the rows and columns supply the block contents in two contrasted types of replication. The block size is thus reduced to the square root of the number of varieties. In a triple lattice a third type of replication is supplied by choosing for the same block varieties having the same letter of a Latin square, which may always be found. Squares, the side of which is an odd number, or a multiple of 4, will also always yield a fourth type of replication, making a quadruple lattice. For prime numbers, and, as appeared later in the discussion, for all powers of primes, a complete set of mutually orthogonal classifications is possible, leading, if one replication of each type is used, to one of the known solutions of the problem of randomized incomplete blocks. We have, however, free choice in the topographical arrangement of the plots, and if the number of varieties is the square of an odd number, such as 11, we may halve the number of complete replications by superimposing pairs of these in a Latin square, so that six 11×11 squares will suffice to give equal precision to all comparisons. This considerable advantage, combined with the high precision to be expected in Latin square designs, makes the scheme as attractive experimentally as it is mathematically elegant.

Mr. W. L. Stevens, of the Galton Laboratory, had a surprise in store, in the form of a demonstration of the fact that for any power of a prime a completely orthogonal square exists. The converse had been asserted, on the basis of an erroneous proof by Wernicke in 1910, but the theorem had appeared to be probably true from the construction in 1936 of completely orthogonalized squares of sides 8 and 9, by Yates and Fisher respectively. It was known to his associates that Mr. Stevens had already established a demonstration for the square of any prime number, but the very simple generalization which he gave had only occurred to him during the week.

Based on the theorem that if s is a power of a prime, a field of s symbols with the corresponding operations of addition, subtraction, multiplication and division can be defined so that the results of these operations fall within the field and are unique, we may define a Latin square by the equation

$$u_L = u_\lambda u_R + u_C \qquad u_\lambda \neq 0_C$$

where u_R, u_C and u_L designate the row, column and letter of any element, and u_λ is arbitrary apart from the restriction that it may not be zero.

Assigning its $s-1$ possible values to u_λ generates $s-1$ Latin squares, and the uniqueness of the solution of the equations, regarded as simultaneous in u_R and u_C

$$u_i = u_\lambda u_R + u_C,$$
$$u_j = u_\mu u_R + u_C,$$

shows that the element having a given letter i in the first square, and a given letter j in the second, is uniquely determined in row and column. Thus any two Latin squares of the set are orthogonal, and the whole constitutes a completely orthogonalized square.

It was interesting to learn that the 9×9 squares obtained in this way are different from one previously given by Yates, so that for the larger squares a multiplicity of completely orthogonal solutions is to be anticipated.

It is much to be regretted that the programme allowed no time for discussion. We should have liked to hear the reaction of many mathematicians in the audience, which remained to the end closely interested.

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Plant Growth Substances

FOLLOWING a project first put forward in 1925, the Committee on Intellectual Cooperation of the League of Nations and the International Council of Scientific Unions have agreed to collaborate in the calling of occasional conferences on well-defined fields. The first of these, on the plant growth substances, or phytohormones, was held in Paris on October 1–2, 1937, under the joint auspices of the C.I.C. and the International Union of Biological Societies.

The report of the conference,* which has just been issued, contains eight contributed papers and discussion. Prof. Kögl (Utrecht) describes the determination of the chemical nature of auxins a and b and the isolation of biotin. Dr. Niels Nielsen (Copenhagen) discusses the substances promoting growth in the fungi, and the difficulties introduced by the varying abilities of organisms to synthesize different members of the group of active substances. The evidence associating the formation and action of auxins with oxidative metabolism is reviewed by Prof. Koningsberger (Utrecht), the phenomenon of bud inhibition and other correlations by Prof. Dostál (Brno) and the relation between the phytohormones and plant tropisms by the chairman, Prof. Boysen

* Etudes et Recherches sur les Phytohormones : Première Réunion organisée en collaboration avec l'Union Internationale des Sciences Biologiques, Paris, 1 et 2 Octobre, 1937. Pp. xiv+125. (Paris : Institut International de Coopération Intellectuelle.)