The Electron as a Vector Wave.

By Prof. C. G. DARWIN, F.R.S.

THE spinning electron of Uhlenbeck and Goudsmit has brilliantly filled up a serious gap in atomic physics, but, while we cannot withhold our admiration from its successes, it is only fair to consider certain defects from which it suffers. When what is required is to double the number of states of the electron, it is at the least generous to introduce three extra degrees of freedom and then make an arbitrary (though not unnatural) assumption which cuts down the triple infinity to two. The electron is in fact given a complete outfit of Eulerian angles, even if it may not be necessary so to express the matter explicitly. Now we regard the electron as the most primitive thing in Nature, and it would therefore be much more satisfactory if the duality could be obtained without such great elaboration. The present communication is an attempt to do this; it is, I think, promising, though falling short of complete success, but as future stages would involve a very large amount of work, it seemed better to expose the theory to criticism at once, in case some serious objection can be made against the whole principle of it.

The above criticism of the spinning electron is perhaps partly aesthetic, but there are others. Thus though dynamical principles have been shown by both Thomas (NATURE, **117**, 514) and Frenkel (*Zeit. f. Phys.*, **37**, 243) to give doublet spectra correctly, yet neither of them has succeeded in casting the result in a rigorous Hamiltonian form, so that all the work on the spinning electron has a 'dressed-up' appearance, lacking in formalism. Again the wave mechanics (in this the matrix mechanics is better) definitely excludes half quantum numbers for the spin, and so would lead to triplets instead of doublets—1, 0, -1 instead of $\frac{1}{2}$, $-\frac{1}{2}$. It is perhaps also not unfair to argue that the quantum mechanics is largely guided by the principle that nothing unobservable is to be explained, so that a theory is to be regarded as suspect, which introduces a large number of higher quantum states of rotation, only to bar them later.

The advent of the wave mechanics must have suggested to many a way out of these difficulties by assimilating the electron to a transverse rather than a longitudinal wave, for this at once provides the number of states with the necessary factor 2. The idea involves difficulty when more than one electron is present; we shall discuss and tentatively meet this difficulty later, but a necessary preliminary is to obtain a system of equations for the single electron. In doing so we are endowing it with an intrinsic duality to which there is no direct analogue in dynamics, so that the only guidance we have is that the equations must be of the wave form (to conform to classical dynamics in the limit), and must be such as to give correctly the structure of doublet spectra. It is scarcely conceivable that the equations should not involve the Schrödinger functions, and this excludes, as a great many trials showed, any wave type built on lines like the electromagnetic equations, for the Schrödinger function will not tolerate a *divergence* relation of any kind. Moreover, such types of wave do not appear to exhibit those qualities of approximate degeneracy which are implied by the Paschen Back effect. It is, however, possible to construct by a much more inductive process a system of waves, of a vector character though not transverse, which completely reproduces the doublet spectra, and then to generalise from this.

The general character of doublet spectra is given by having two *dependent* variables, each of which nearly satisfies the same equation, say $Df=0$ where D, depending on x, y, z, t, is the operator in Schrödinger's equation. Let α, β, γ, δ be small perturbing operators in x, y, z, t, and solve the equations

$$\left.\begin{aligned} Df+\alpha f+\beta g&=0 \\ Dg+\gamma f+\delta g&=0 \end{aligned}\right\}, \qquad (1)$$

Near any characteristic W_n of $Df=0$ there are then two solutions of (1) with just the right type of degeneracy. Our task is, therefore, to discover forms of α . . . δ which will give the observed values of W for doublet spectra, and this is not so hard as might appear at first sight.

The *terms* of a doublet spectrum are very conveniently epitomised as follows. We take $k=0$, 1, 2 . . . for s, p, d . . . and $-k \leq m \leq k$, m being integral. For all values of m solve the following equations for $\overline{W}$:

$$\left.\begin{aligned} a_m[\overline{W}-\rho m-\omega(m+1)]-b_{m+1}\rho(k+m+1)&=0 \\ b_m[\overline{W}+\rho m-\omega(m-1)]-a_{m-1}\rho(k-m+1)&=0 \end{aligned}\right\}, \qquad (2)$$

where ρ, ω are the Landé constant for doublet separation and the normal Zeeman effect in energy units. It will be found that the solutions give $\overline{W}$ as the distances of all the Zeeman levels from the centre of gravity of the terms at all strengths of magnetic field. (The equations give two end solutions of different form from the others, and these give the two extra members for one component of the doublet.) The above equations were found by solving the complete problem of the spinning electron and then adjusting the constants by trial; a_m and b_m are then the coefficients of a spherical harmonic for the two directions of spin. We shall not further inquire into their meanings; when suitably normalised they will of course be connected with intensity, but we need feel little doubt that intensities will come right on practically any theory.

By trial with $k=1$ and 2 it is easy to construct the operators α . . . δ and these are then found to work for all cases. The equations (1) are:

$$\left.\begin{aligned} \left(D-\frac{2\pi eH}{ch}\right)f+\frac{1}{2}\frac{Ne^2}{mc^2}\frac{1}{r^3}(-iR_1g-R_2g+iR_3f)&=0 \\ \left(D+\frac{2\pi eH}{ch}\right)g+\frac{1}{2}\frac{Ne^2}{mc^2}\frac{1}{r^3}(-iR_1f+R_2f-iR_3g)&=0 \end{aligned}\right\}, (3)$$

In these D is the Schrödinger operator, written in the form of Dirac (*Proc. R. S.*, **112**, 661) with the view of relativity generalisation:

$$D=\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\left(\frac{2\pi mc}{h}\right)^2+2\cdot\frac{2\pi i}{c^2h}\frac{Ne^2}{r}\frac{\partial}{\partial t}+\frac{2\pi ieH}{ch}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)+\left(\frac{2\pi Ne^2}{ch}\right)^2\frac{1}{r^2}$$

and

$$R_1=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y},\ R_2=z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z},\ R_3=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}.$$

N is the atomic number of the nucleus, and we have reversed Heisenberg's process of deriving Landé's doublet formula from the mean of $1/r^3$. As it will be needed later, we here note that with the present formulation the time occurs in f and g in the form $exp - i\frac{2\pi}{h}(mc^2+W)t$. In spite of their appearance the equations (3) are symmetrical in x, y, z, when $H=0$.

The presence of coefficients Nex/r^3, etc., is a strong invitation for us to generalise, since they are simply the electric forces. Moreover, this generalisation is required if we are to show that the anomalous Zeeman effect is the same phenomenon as the doublet effect. We first split the two equations into four. Multiply by arbitrary constants a, b and add. Then do the same with $ia, -ib$; $-b, a$; ib, ia. We call the quantities $af+bg$, etc., X_1, X_2, X_3, X_4. (The process is nearly, but not quite, the same as taking real and imaginary parts of (3).) The equations then become:

$$\left.\begin{aligned}DX_1-U_1X_4-U_2X_3+U_3X_2=0\\DX_2-U_2X_4-U_3X_1+U_1X_3=0\\DX_3-U_3X_4-U_1X_2+U_2X_1=0\\DX_4+U_1X_1+U_2X_2+U_3X_3=0\end{aligned}\right\},\qquad(4)$$

where

$$U_1=\frac{1}{2}\frac{e}{mc^2}\left(E_y\frac{\partial}{\partial z}-E_z\frac{\partial}{\partial y}\right),$$

$$U_3=\frac{1}{2}\frac{e}{mc^2}\left(E_x\frac{\partial}{\partial y}-E_y\frac{\partial}{\partial x}\right)+i\frac{2\pi e}{ch}H_z.$$

These equations, which are really only two, are now in vector form for space transformations, regarding X_4 as a scalar and X_1, X_2, X_3 as a vector. We can therefore take the magnetic force in any direction by adding on to U_1, U_2 terms like the last in U_3. It remains to apply the relativity transformation. The first point to observe is that (4) is not in space-*time* tensor form. To make it so we must use the fact that with sufficient approximation $-i\frac{2\pi}{h}mc^2=\frac{\partial}{\partial t}$. Remembering that E_x is the 14 component of the force tensor, this shows that the equations must be put in the form

$$\frac{\partial}{\partial t}DX_1-V_1X_4-V_2X_3+V_3X_2=0,\text{ etc.}$$

where

$$V_1=-i\frac{\pi e}{h}\left(E_y\frac{\partial}{\partial z}-E_z\frac{\partial}{\partial y}\right)+i\frac{2\pi e}{ch}H_x\frac{\partial}{\partial t},\text{ etc.},$$

This is now *dimensionally* a possible tensor equation, but is not in fact covariant for space-time transformations. It is necessary to take the electric terms in V to be twice as large as they really are; this is exactly the trouble of Uhlenbeck and Goudsmit. We shall discuss it later, for the present simply doubling the first factor in V_1, etc. Now write $x_1x_2x_3x_4$ for x, y, z, ict, and $\phi_1, \phi_2, \phi_3, -i\phi_4$ for the vector and static potentials; also for the six forces put $F_{12}=\frac{\partial\phi_2}{\partial x_1}-\frac{\partial\phi_1}{\partial x_2}$, etc. Remembering that $\partial/\partial x_4$ is much larger than $\partial/\partial x_1$, etc., and that H is much smaller than E, we add on certain insensible terms and obtain as our final equations

$$\left.\begin{aligned}T_4X_1-T_1X_4-T_2X_3+T_3X_2=0\\T_4X_2-T_2X_4-T_3X_1+T_1X_3=0\\T_4X_3-T_3X_4-T_1X_2+T_2X_1=0\\T_4X_4+T_1X_1+T_2X_2+T_3X_3=0\end{aligned}\right\},\qquad(5)$$

where

$$T_1=\frac{\partial}{\partial x_1}\left\{\Sigma_\alpha\left(\frac{\partial}{\partial x_\alpha}+i\frac{2\pi e}{ch}\phi_\alpha\right)^2-\left(\frac{2\pi mc}{h}\right)^2\right\}+i\frac{2\pi e}{ch}\left(F_{23}\frac{\partial}{\partial x_4}+F_{34}\frac{\partial}{\partial x_2}+F_{42}\frac{\partial}{\partial x_3}\right)$$

On the present view, apart from the introduced factor 2, these equations constitute the ultimate dynamics of a single electron. It will not alter the observed values, and will perhaps fit better with future generalisations if throughout (5) the operators $\partial/\partial x_1$, etc., are replaced by $\frac{\partial}{\partial x_1}+i\frac{2\pi e}{ch}\phi_1$, etc., as they are in D.

The first three equations of (5) are antisymmetric tensors of the second rank and the last is scalar, but the variables can be permuted according to the following scheme, so that any one of the four equations is the scalar:

$$\begin{matrix}X_1\\X_2\\X_3\\X_4\end{matrix}\rightarrow\begin{Bmatrix}X_4\\X_3\\-X_2\\-X_1\end{Bmatrix}\rightarrow\begin{Bmatrix}-X_3\\X_4\\X_1\\-X_2\end{Bmatrix}\rightarrow\begin{Bmatrix}X_2\\-X_1\\X_4\\-X_3\end{Bmatrix},\qquad(6)$$

The existence of these permutations means that to regard the X's as a vector is an unnecessary restriction, to which it may not always be convenient to submit. For example, by allowing a different rule of transformation the theory could be put in terms of four purely real quantities.

We must next consider how the solution will go when more than one electron is present. This is a most important matter, for the replacement of independent by dependent variables might entirely alter the counting of the number of solutions. We shall omit relativity considerations and so may use (4) instead of (5). At first sight the most natural extension would be to regard X_1, etc., as functions of two sets of x, y, z; thus the first equation of (4) would become

$$(D+D')X_1-(U_1+U'_1)X_4-(U_2+U'_2)X_3+(U_3+U'_3)X_2=0.$$

The effect would be to double the number of solutions, but it would seem that they would sort out as $2+2$, not $3+1$ as is required. We need a process which will give $4+4$ solutions, reduced to $3+1$ by Heisenberg's resonance principle (*Zeit. f. Phys.*, **38**, 411). This could probably not be done for most vector waves, but the permutations (6) suggest a way out; for in combining the two sets of variables there is no particular reason to select the same permutation of both. The actual number of possible selections is easiest seen from (3), where in adding the effects together we can interchange the meanings of f and g. We shall in this way get two systems of equations each with four solutions, just the number required. The question of two or more electrons is, I think, the most serious difficulty for the present theory, and this is only intended as an incomplete and tentative suggestion as to how it may be met.

Finally, we must consider the factor 2 which had to be introduced to obtain the tensor form. This is evidently the original difficulty of Uhlenbeck and Goudsmit. It was removed by Thomas, who showed that a rigid body when accelerating exhibits a sort of rotation on account of the kinematics of relativity. This brilliant explanation resolves the disagreement, but it really imports a foreign idea into mechanics; indeed relativity and rotation do not take at all kindly to one another, and it is not surprising that no formal Hamiltonian method has been found to cover what is really a blemish in geometry rather than dynamics. As we here have nothing corresponding very exactly to velocity, we cannot use the same type of argument. But regarding the matter from a more abstract point of view, may we not perhaps draw an inference from the fact that our work has forced us to have equations of the third degree? Relativity is essentially a point theory and is governed by a quadratic form. To a first approximation motion is controlled by this form, and the associated wave equations are of the second degree. Now we have seen that the actual wave equations, though approximately of the second degree, are more accurately of the third. Taking the analogy back, may we not conjecture that the quadratic form of space-time wants amplifying in some way (I fear the idea is quite vague) by terms of the third degree, and that the reason why Thomas and Frenkel did not obtain a formal Hamiltonian is because 'quadratic' dynamics is only an approximation, which cannot be perfectly represented by importing into relativity theory the foreign idea of a rigid rotating body. But this speculation is too indeterminate to pursue further at present.

Nerves and Muscles: How we Feel and Move.

By Prof. A. V. Hill, F.R.S.

It is not an easy task to give the Christmas Lectures at the Royal Institution, especially on a subject like physiology, in which experiments are much more difficult than in some others, at least when required to work without fail, to be intelligible and not to cause offence. The difficulty is increased by the fact that one's audience does not consist simply, in the words of the Royal Institution, "of juveniles between the ages of 10 and 16"; one finds oneself addressing distinguished adults in many walks of life, and not only these, but also through the publicity given to these lectures in the press, a considerable proportion of the people of Great Britain. One must reflect, however, in preparing them that they are intended for 'juveniles,' and that if others dare to come they do so at their own risk and will be assumed to be 'juveniles' for the purpose of the lectures. It is true, as a matter of fact, that most lecturers are apt to pitch their lectures at a level much too high for their audience, and that by attempting to make them interesting to a child of thirteen they may well succeed in absorbing the attention even of adults comparatively expert in the matter with which the lecture deals. We all like to have things put to us simply, and no lecturer to a juvenile auditory should pay any attention whatever to people above the age of sixteen.

The Christmas lectures at the Royal Institution are intended to include a considerable number of demonstrations and experiments, and without the use of live animals experiments on the functional working of the body are impossible. Two fortunate facts, however, make it practicable to give suitable phsyiological demonstrations at the Royal Institution: (1) that the isolated organs and tissues of dead cold-blooded animals will continue to function for considerable periods after removal from the animal; and (2) that in recent years a great variety of perfectly good physiological experiments, often of precise quantitative as well as qualitative significance, have become possible upon men; and if upon men, then also upon children.

The fact that isolated tissues of animals will function for a long time after removal, although a commonplace of phsyiology, is not generally realised, and in itself is a matter of entrancing interest to those who come upon it for the first time: it awakens all kinds of questions, scientific and philosophical, which not only a 'juvenile' but also an adult urgently desires to see answered. The beating of an isolated heart, which will continue for hours or days after removal from its previous owner, provided that it be properly treated, is an extraordinary eye-opener to most people previously ignorant of phsyiology. There are few physiologists who do not remember the astonishment and interest which was aroused in them by their first sight of this phenomenon, or of the reflex movements of a frog whose head has been cut off, or of the contraction of a muscle preserved, sometimes for days, in salt solution, or of the electric current produced by an isolated nerve, or of the many other strange things that happen to little bits of tissue removed from the ordinary environment which their previous owner supplied. It is such things that bring physiology first into the region where exact experiments can