Summary of the Theory of Relativity.

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I. BREAKDOWN OF OLDER THEORIES.—The older electromagnetic theory of moving bodies did not agree with experiment, or even with itself. For example, the theory of a magnet moving in a straight line towards a fixed conductor gave results quite different from those of the theory of a conductor moving in a straight line with the same velocity towards a fixed magnet. Yet experiment showed that the results should be the same, depending only on the *relative* velocity. Again, the æther was assumed to be at the same time quite unaffected by the earth's motion (to explain aberration), partly affected (to explain Fizeau's water-tube experiment), and entirely affected (to explain the experiments of Michelson and Morley, Lodge, Rowland, Rayleigh and Brace, Trouton and Noble, and others).

II. FUNDAMENTAL ASSUMPTIONS OF EINSTEIN'S RESTRICTED THEORY (1905).—This takes over Maxwell's theory so far as it applies to bodies at rest relative to the earth and deals with other systems by the two following assumptions:

(1) All electrodynamical and optical equations which hold for a system S hold also for another system S′ which, relative to S, moves with uniform velocity v in a straight line.

(2) Light is propagated in a vacuum with a velocity c which appears the same for observers in S and S′.

Kinematical deductions from these assumptions.—These imply that the measures of time and space in S and S′ must be such that

$$x^2+y^2+z^2-c^2t^2=x'^2+y'^2+z'^2-c^2t'^2,$$

from which, taking the corresponding axes in each system to be parallel and the relative velocity to be along Ox (or Ox'), we can prove that

$$x'=\beta(x-vt),\ y'=y,\ z'=z,\ t'=\beta\left(t-\frac{vx}{c^2}\right)$$

$$\text{where } \beta=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}. \quad \text{(A)};$$

hence two observers, one in S and one in S′, will *each* imagine

(i.) that a rod along Ox (or Ox') in the other's system has contracted in the ratio $\beta:1$;

(ii.) that the other's clocks (supposed controlled by light signals) lose, taking β seconds instead of 1 for a beat;

(iii.) that the events which the other takes as simultaneous are not so.

What they will agree about is the velocity of light, c, their own relative speed, and the *interval* between two sets of values, x, y, z, t, for two events, this *interval* being defined as

$$\sqrt{\{c^2(t_2-t_1)^2-(x_2-x_1)^2-(y_2-y_1)^2-(z_2-z_1)^2\}},$$

which may be written,

$$\sqrt{\{c^2dt^2-dx^2-dy^2-dz^2\}}.$$

It is generally denoted by ds.

From equations (A) $\dfrac{dx'}{dt'}=\dfrac{\dfrac{dx}{dt}-v}{1-\dfrac{v}{c^2}\dfrac{dx}{dt}}$, so that if the velocity of the body moving along Ox (or Ox') is V in the system S and V′ in the system S′

$$V'=\frac{V-v}{1-\dfrac{vV}{c^2}}, \text{ or } V=\frac{V'+v}{1+\dfrac{vV'}{c^2}}.$$

This is confirmed by Fizeau's water-tube experiment, and (it is claimed) by Majorana's moving mirror experiment. From this formula we see that by combining two velocities V′ and v, each of which is smaller than c, we obtain a velocity V which is always smaller than c. (The statement that "*no velocity can exceed c*" is too sweeping; the velocity of light in a thin metal prism exceeds c.)

Electrodynamical deductions from these assumptions.—Transforming Maxwell's equations for free space in which electrons move with velocity V along Ox we get from assumption (1) and equations (A) that

$$\left.\begin{aligned} E'_x&=E_x, & H'_x&=H_x \\ E'_y&=\beta\left(E_y-\frac{v}{c}H_z\right), & H'_y&=\beta\left(H_y+\frac{v}{c}E_z\right) \\ E'_z&=\beta\left(E_z+\frac{v}{c}H_y\right), & H'_z&=\beta\left(H_z-\frac{v}{c}E_y\right) \\ \rho'&=\beta\rho\left(1-\frac{vV}{c^2}\right). \end{aligned}\right\} \quad \text{(B)}.$$

The expression for ρ' gives the remarkable result that the charge on an electron appears the same in both systems. From these we can deduce:

(i.) Doppler's effect in the modified form—

$$f'=f\left(\frac{1-\dfrac{v}{c}}{1+\dfrac{v}{c}}\right)^{\frac{1}{2}},$$ where v is the relative velocity in the line of sight, f and f' the frequencies;

(ii.) a modified law of aberration;

(iii.) the force exerted by light on a moving mirror;

(iv.) the electric and magnetic fields due to a uniformly moving electron.

The differences between these forms and those given by older theories are too small to be detected by experiment.

Dynamics of an electron (slowly accelerated).—With the *additional assumption* that every electron has a constant m associated with it, such that *force = m × acceleration* at the instant when the electron is at rest in the system of co-ordinates used (and only at that instant), we deduce that in any other system the equations of motion are

$$\left.\begin{aligned} m\beta^3\frac{d^2x}{dt^2}&=eE_x, \\ m\beta\frac{d^2y}{dt^2}&=e\left(E_y-\frac{v}{c}H_z\right), \\ m\beta\frac{d^2z}{dt^2}&=e\left(E_z+\frac{v}{c}H_y\right), \end{aligned}\right\}$$

where e is the charge on the electron and the axis of x is taken in the direction of its velocity v. The second and third of these equations are confirmed by Bucherer's experiments.

If, with Lorentz, we take the right-hand sides as the components of the force, and retain the old law *force = mass × acceleration*, we find it necessary to speak of a longitudinal mass $m\beta^3$ and a transverse mass $m\beta$.

But we may rewrite the left-hand sides in the symmetrical form $\dfrac{d}{dt}\left(m\beta\dfrac{dx}{dt}\right)$, $\dfrac{d}{dt}\left(m\beta\dfrac{dy}{dt}\right)$, and $\dfrac{d}{dt}\left(m\beta\dfrac{dz}{dt}\right)$

This suggests the definitions:

mass (M) = mass at low speeds × β (both for longitudinal and transverse mass);

momentum = mass × velocity;

force = rate of change of momentum.

Defining work in the usual way from force and displacement, we can further deduce:

Work done on an electron = increase of its kinetic energy, provided that kinetic energy is defined as Mc^2 + a constant = $m\beta c^2$ + a constant.

If we take the constant equal to $-mc^2$, this new definition reduces to $\frac{1}{2}mv^2$ approximately for small values of v/c. From Maxwell's equations we can derive four relations for an isolated system of electrons which may be interpreted as the conservation of momentum and of energy, provided that the momentum and energy of the electrons are defined as above, and that the momentum and energy of the field are included, the momentum of the field per unit volume being defined as Π/c^2, where Π is Poynting's vector. Observations on the spectral lines of hydrogen, and Guye and Lavanchy's experiments on cathode rays, confirm these results.

III. FUNDAMENTAL ASSUMPTIONS OF EINSTEIN'S GENERALISED THEORY (1915).—(1) For an infinitely small region of space and time, axes may be chosen so that the restricted theory is true in that region. This implies that for two events there exists a certain absolute quantity, the interval ds, which, by a *suitable* choice of co-ordinates, may be expressed as before, but which in a general system of co-ordinates, $x_1\ x_2\ x_3\ x_4$ (these being arbitrary functions of $x\,y\,z\,t$), take the form $\sqrt{(\Sigma_r \Sigma_s g_{rs} dx_r dx_s)}$, where r and s take all values from 1 to 4, and the g's are functions of $x_1\ x_2\ x_3\ x_4$.

(2) All physical laws must be expressible by means of equations which are valid for all co-ordinate systems. That is to say, the equations are covariant, or unaltered in form, for the most general transformation (not necessarily linear). Newton's law of gravitation and all other laws that do not satisfy this condition are to be modified so as to conform with it.

(3) *The Principle of Equivalence.*—A gravitational field of force at a point or infinitely small region is exactly equivalent to a field of force introduced by a transformation of the co-ordinates of reference, so that by no possible experiment can we distinguish between them. (Eddington pointed out that the assumption is made for phenomena which depend on the g's and their first differential coefficients, and in general it will *not* apply to those involving second differential coefficients.)

(4) The path of a particle in a gravitational field is such that $\delta\int ds = 0$. (For the case when there is no gravitation this reduces to Newton's first law of motion.) This assumption reduces particle dynamics to something like the geometry of geodesics on surfaces, except that we have four independent variables instead of two.

(5) Although the coefficients in the expression for ds^2 are capable of infinitely many forms, according to the system of co-ordinates used (just as in measurements on a surface the square of the shortest distance on the surface between two points can be similarly expressed in many forms corresponding to the choice of the independent variables), yet these g's are not quite arbitrary functions of the co-ordinates, but satisfy a set of partial differential equations (analogous to those which for a surface express intrinsic properties of that surface). These differential equations are *assumed* to be of a certain particular form, known as those expressing the vanishing of the *contracted Riemann-Christoffel* tensor. (A tensor may roughly be defined as a generalised vector. If all its components vanish in one system of co-ordinates, they all vanish in any other system.) This assumption is not quite as arbitrary as it looks, for it is the second simplest set which is of the covariant form required by assumption (2). The simplest set of all corresponds to the absence of any gravitational field.

(6) The energy of a gravitational field exerts gravitating action just like ordinary masses. This assumption leads to equations which may be interpreted as implying the conservation of momentum and energy, including contributions due to the gravitational field (and to the electromagnetic if present).

Mathematical Deductions from these Assumptions.—(*a*) *Formulæ for the Interval.*—By solving the differential equations the g's may be obtained. The number of solutions is infinite. For a single heavy mass, choosing the units so that c and the gravitational constant are unity,

Schwarzschild gave

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

F. W. Hill and G. B. Jeffery gave

$$ds^2 = \frac{\left(1 - \frac{m}{2r}\right)^2}{\left(1 + \frac{m}{2r}\right)^2}dt^2 - \left(1 + \frac{m}{2r}\right)^4(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2).$$

and Painlevé has given a great variety.

(*b*) *Perihelion of Mercury.*—From any of these forms and assumption (4) we can by the Calculus of Variations determine the orbit of a planet. The orbits so deduced differ very little from those calculated on the Newtonian laws. The only difference big enough to be observed is that for Mercury. Leverrier estimated that the older theory differed from observation by about 43″ per hundred years. Einstein's theory accounts for these 43″. (But Grossmann (1922) has recalculated the old discrepancy as 38″, not 43″.)

(*c*) *Deflection of Ray of Light by Sun's Gravitational Field.*—The rays should be slightly curved, as if the gravitational field round the sun were a converging lens, thus making stars on opposite sides of the sun appear farther apart than when the sun is in another part of the sky. The result of the measurements made during the solar eclipse of May 29, 1919, agreed very closely with Einstein's predictions. This is strong evidence in support of Einstein's modification of the Newtonian law, as on the old law the deflection should be only half the amount predicted by Einstein and actually observed.

(*d*) *Spectral Shift.*—Einstein believes that the formula for ds^2 implies that the spectral lines in the light coming to us from the surfaces of big stars should appear shifted towards the red end of the spectrum. Eddington and others think it possible that this argument may be founded on an assumption which may be rejected while the rest of the relativity theory is retained. Grebe and Bachem (Bonn) claim to have observed the predicted effect, and so do Perot and Buisson and Fabry; St. John claims to have shown that it does not occur, but his results have been doubted. The experimental difficulties are enormous.

(*e*) *Apparent Contraction of a Rod placed radially in a Gravitational Field.*—Einstein deduces this from the formula for ds^2 and also deduces that there is no such tangential effect. Painlevé (1921) strongly objects to these deductions and points out that by taking other forms of ds^2 we can reject these conclusions, while retaining all the verifiable results of the theory. If Einstein's views are correct, Euclidean geometry (*e.g.* Pythagoras's theorem) is not exactly true for measurements made in a gravitational field. It will be replaced by Riemann's geometry

IV. EINSTEIN'S COSMOLOGICAL THEORY (1917).—The leading feature of this is that our universe, *as measured by material rods or light rays*, is finite, so that a ray of light will never get more than a certain distance from its starting-point. However, he is willing to admit that other universes may exist outside this limit, but such that their light can never meet ours. Eddington and others regard this theory rather unfavourably.

V. EINSTEIN'S VIEWS ON THE ÆTHER (1920).—Space is endowed with physical qualities. In this sense, therefore, there exists an "æther." Without it there would be no propagation of light. But this æther may not be thought of as endowed with the physical properties of material media. It must not be considered as either fixed or moving. No explicit use of any conception of the æther is made in the theory of relativity. It is difficult to see what use could be made of the above views, which are chiefly negative. The phenomena of the gyroscope and Foucault's pendulum (and Sagnac's optical experiment), which on the Newtonian ideas are attributed to absolute space, are attributed by relativists to the æther or the effects of the fixed stars—which is rather unconvincing.

VI. WEYL'S EXTENDED THEORY (1918).—Whereas Einstein's interval depends only upon gravitational phenomena (although Maxwell's equations and all electromagnetic effects fit into the framework thus constructed), Weyl assumes that the length of the measuring rod depends upon the route it has taken in the neighbourhood of electromagnetic fields. When these are present, the interval is no longer a definite quantity (thus weakening the argument for the spectral shift). This theory accounts for Maxwell's equations and introduces Einstein's cosmological term in a natural way, and adds the law of conservation of electricity to those of conservation of momentum and energy. On the other hand, it introduces great complexity into geometry and appears to imply the impossibility of metrology, beyond a certain—very high—degree of accuracy. There is no experimental confirmation. Einstein does not accept it. Eddington (1921) has generalised Weyl's mathematics, but says, "Einstein's postulates and deductions are exact. The natural geometry of the world . . . is the geometry of Riemann and Einstein, not Weyl's generalised geometry or mine."

VII. PAINLEVÉ'S SEMI-EINSTEINIAN THEORY OF GRAVITATION (1922).—This retains Euclidean geometry and the old ideas about space and time. By axioms which are somewhat similar to those of Einstein, but which make no reference to the restricted theory, Schwarzschild's form of ds^2 and the verified astronomical results are obtained.

Kitchen Ranges.

THERE is probably no more difficult problem presented to the heating engineer than the kitchen range. So complicated is it that it would appear that no single appliance could possibly be constructed to suit every house or even any large number of houses, and that each installation would have to be adapted to the requirements of the special household. For example, a working-man's cottage usually requires only one fire, which, in the absence of a gas cooker, must satisfy the quadruple duty of heating the room, the oven, the hot-plate and the water, whereas a better class of house might use, and with greater economy, a gas cooker and a coke boiler for the supply of hot water and radiators. Then, again, in an ordinary household, cooking is an operation occupying two or three hours per day only, while hot water is likely to be required at any moment throughout the day. Heating of the rooms is required continuously all day in winter, but not at all in summer. The inevitable consequence of such an intermittent demand is a low efficiency.

We have before us two important pamphlets embodying the researches of Dr. Margaret Fishenden and Mr. A. H. Barker carried out under the auspices of the Fuel Research Board.[1] Dr. Fishenden has restricted her investigation to the comparative efficiency of ranges fired with ordinary bituminous coal and those heated with the special coke cakes (low temperature coke) produced by the Fuel Research Station at E. Greenwich. She finds that low temperature coke yields a greater proportion of total heat for radiation or for water heating than bituminous coal, while for oven heating the coke compares less favourably with coal, the advantage of coke being largely due to radiation effects. She finds, moreover, that in an open kitchen range with back boiler about 17 per cent. of the heat of the coal is used for hot water, and in modern designs it varied from 13 to 19 per cent., a result rather higher than that found by Mr Barker.

It is unfortunate that Dr. Fishenden's experiments do not include ordinary coke, as the low temperature coke prepared by the Fuel Research Board is a commodity not yet on the market and unlikely to appear there, as it is obviously too costly to compete at present with either coal or coke. The report of Mr. Barker (who is lecturer on heating and ventilating engineering at University College, London) deals in a very comprehensive fashion with the whole subject of kitchen ranges, and the results of a large number of practical tests on old and new designs using coal, coke, and gas as sources of fuel. The introduction to the report contains the following statement: "In the design of British cooking ranges, attention has hitherto been mainly devoted to securing cheapness of construction and convenience of use. Economy in fuel consumption has only played a minor part in determining the different types in use. The shortage and high price of coal have, however, emphasized the necessity for fuel economy and, consequently, of an examination of the efficiency of British kitchen ranges. . . . The strong prejudice in favour of an open-fronted fire appears to be peculiar to this country. In most other countries a cooking range fire is usually closed. . . . In view, therefore, of the scarcity and high price of coal at the present time, it appears to be a matter for serious consideration whether steps should not be taken to encourage the more general adoption in this country of ranges which are more economical in fuel consumption than those of ordinary British design.'

In his general summary Mr. Barker has arrived at the following conclusions: that the general efficiency of all ranges on the market at the present time is low, the actual oven efficiency ranging from 0.75 to 5 per cent., the usual being about 2 per cent., that of the hot water supply from 7 to 17 per cent. or usually 11 per cent., and the hot plate from 1 to 12 per cent. or generally below 6 per cent. He estimates that the modern type of range wastes 85 per cent. of the fuel in heating the air of the kitchen (about 30 per cent.), by absorption in the brickwork (about 30 per cent.), and lost in the flue gases (about 25 per cent.). Economy may be effected by not setting ranges in brickwork, by preventing leakage of cold air into the furnace and flues, and by doing away with the hot-plate or covering it when not in use, and also the oven door, with non-conducting material. He admits, however, that these losses are unavoidable if the present convenience and cheapness of the ordinary range are to be retained and one fire made to serve so many different purposes. But if the efficiency is considered irrespective of convenience, cheapness,

[1] (1) The Efficiency of Low Temperature Coke in Domestic Appliances, by Dr. Margaret W. Fishenden. Fuel Research Board, Technical Paper No. 3. London: H.M. Stationery Office, 1922. 9d. net.
(2) Tests on Ranges and Cooking Appliances, by A. H. Barker. Fuel Research Board, Special Report No. 4. London: H.M. Stationery Office, 1922. 2s. 6d. net.