LETTERS TO THE EDITOR.

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Magic Squares of the Fifth Order.

IN the interesting discourse reproduced in your issue of March 13 (p. 447), there is a statement that the number of magic squares of order 5 exceeds 60,000. Major MacMahon informs me that he gave these figures on the authority of Rouse Ball's "Mathematical Recreations." The statement is not wrong, but viewed as a minimum limit it may be largely exceeded. I have recently investigated the total number of squares of this order, which have the additional property that the nine numbers in the heart of the square also form a magic—the well-known "bordered squares." Fig. 1 is an example. The square itself is magic in rows, columns and diagonals, and the nine numbers in the central square show like properties.

It is easy to see (1) that the numbers in the central square must consist of three arithmetical progressions with a like common difference; (2) that the first terms of each progression must also be in arithmetical progression; and (3) that the mean number (13) must always occupy the central cell. Hence it follows that (excluding the central number) the eight numbers in the heart must consist of four pairs of complementary numbers (i.e., pairs whose sum is 26, or twice the mean), and also that the two smallest numbers being known, then all the others are known. If, for example, the two lowest numbers in the heart are a , $a+k$, then the nine numbers in the centre, taken in numerical order, must be a , $a+k$, $a+2k$; $13-k$, 13, $13+k$; $26-2k-a$, $26-k-a$, $26-a$. It is now easy to determine in how many ways we may choose the nine numbers for

2	25	23	10	5
14	11	22	6	12
19	8	13	18	7
9	20	4	15	17
21	1	3	16	24

FIG. 1.

5	25	24	4	7
11	10	23	6	15
12	9	13	17	14
18	20	3	16	8
19	1	2	22	21

FIG. 2.

the centre on the usual assumption that we are restricted to the first 25 natural numbers without repetition or omission. The total is 26, as shown in the first column of the table below, where the figures in brackets denote the two lowest numbers in the heart.

Type.	Number of subtypes.	Type.	Number of subtypes.
A. (1, 2) ...	28	N. (4, 5) ...	30
B. (1, 3) ...	31	O. (4, 6) ...	23
C. (1, 4) ...	18	P. (4, 8) ...	30
D. (1, 6) ...	21	Q. (5, 6) ...	16
E. (2, 3) ...	30	R. (5, 7) ...	20
F. (2, 4) ...	30	S. (5, 8) ...	20
G. (2, 5) ...	26	T. (6, 7) ...	19
H. (2, 6) ...	29	U. (6, 8) ...	18
I. (2, 7) ...	23	V. (6, 9) ...	28
J. (3, 4) ...	23	W. (7, 8) ...	19
K. (3, 5) ...	23	X. (8, 9) ...	29
L. (3, 6) ...	13	Y. (8, 10) ...	18
M. (3, 7) ...	27	Z. (9, 10) ...	10
TOTAL ...	602		

These we may call the 26 main types of bordered squares of the fifth order. The 16 remaining numbers have evidently to be arranged in the four borders in such a way that complementary pairs are opposed at the opposite ends of rows, columns, or diagonals, with the additional condition that the four borders must each sum 65. Now when we deal a pair of complementary numbers into the top and bottom rows,

we clearly give to those rows a difference equal to the difference of the complementary pair. If we call this a "complementary difference," then the numbers in the two rows must be so related that the sum of two complementary differences must equal the sum of the other three, and a similar relation must hold between the two lateral borders. Suppose we are dealing with type L. (3, 6). The numbers for the centre are 3, 6, 9; 10, 13, 16; 17, 20, 23. If now we arrange the remaining numbers in two columns, with complementaries adjacent, and form a third column of half differences, we have:—

		Half Difference.
1	25	12
2	24	11
4	22	9
5	21	8
7	19	6
8	18	5
11	15	2
12	14	1

It is then necessary to form two equations from the column of half-differences of the form $a+b=c+d+e$, and so related that two and only two numbers shall be common to both equations, and that these two shall be on opposite sides of the equality sign in one equation, and on the same side in the other. In the case of type L. we can do this in thirteen different ways, as shown below, the italicised figures denoting the corner pairs:—

$$\begin{array}{ll}
 12+11=9+8+6 \} a. & 12+8=9+6+5 \} b. \\
 6+5=8+2+1 \} & 8+6=11+2+1 \} \\
 12+6=9+8+1 \} c. & 12+6=9+8+1 \} d. \\
 11+5=8+6+2 \} & 9+5=11+2+1 \} \\
 12+5=9+6+2 \} e. & 12+5=9+6+2 \} f. \\
 11+8=12+6+1 \} & 8+6=11+2+1 \} \\
 12+2=8+5+1 \} g. & 12+1=6+5+2 \} h. \\
 11+5=9+6+1 \} & 9+8=11+5+1 \} \\
 11+9=12+6+2 \} i. & 11+9=12+6+2 \} j. \\
 9+8=11+5+1 \} & 6+5=8+2+1 \} \\
 11+8=12+6+1 \} k. & 11+8=12+5+2 \} l. \\
 9+5=11+2+1 \} & 11+5=9+6+1 \} \\
 11+5=8+6+2 \} m. & 9+6=12+2+1 \}
 \end{array}$$

Each of these 13 pairs yield a subtype under type L. For example, take the first pair. The first equation tells us that the greater of the two pairs of numbers whose half-differences are 12 and 11 can be associated in one border with the lesser of the three pairs whose half-differences are 9, 8 and 6, the complements, of course, being opposed in the opposite border. The other equation gives similar information regarding the other two opposed borders. The centre can be arranged at once by following the order of the ordinary magic of the third degree. The result is shown in Fig. 2.

The table above gives the number of subtypes for each main type, the total being 602. The result has been verified by another worker independently.

It is now easy to calculate how many bordered magics of the fifth order exist. The centre for a given subtype we know can be arranged in one way only if reversions and reflections are not reckoned as different; in eight ways if they are so reckoned. Consider the top row in any one of the 602 subtypes. For the left-hand corner we have four choices, for the next cell six, and for the third cell two. This leaves three choices for the second cell of the left-hand border and two for the third. The numbers in all the remaining border cells are now known. There are, therefore, $4 \times 6 \times 2 \times 3 \times 2 = 288$ ways of arranging the borders for each sub-type. The total number of squares is thus $602 \times 288 \times 8$ if reversions and reflections are reckoned as different; $602 \times 288 = 173,376$ if they are not so reckoned.

When we bear in mind that this is the number of magics of a restricted type, it is clear that the number of magics of the fifth order must largely exceed this total; indeed, everything suggests that the totality of magics of the fifth order is more than twice as great as the above result.

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Haywards Heath, March 15.