About a certain Class of Curved Lines in Space of n Manifoldness.

THE class of curves to be considered is defined by the following property: A curve of that class situated in plane space of n manifoldness is cut by a S_{n-1} in n (different or coinciding) points. In the plane it is therefore a conic, and in space a twisted cubic.

If through $n - 1$ of its points a pencil of S_{n-1} is drawn, then each element of that pencil cuts out of the curve *one* additional point, and has with a straight line *one* point in common. The coordinates of the curve must therefore be expressible as rational functions of one parameter. If any fixed pyramid A_1, A_2, . . . A_{n+1} is accepted as pyramid of reference, then any point P of the curve

$$(\Sigma\chi_i) \,.\, P = \chi_1 A_1 + \; . \; . \; . + \chi_{n+1}A_{n+1},$$

where the χ_i are the homogeneous coordinates of P; and it follows

$$\chi_i = R_1(\lambda, \mu) \; . \; . \; . \; \chi_i = R_i(\lambda, \mu),$$

where the R_i are homogeneous and integer functions of the λ, μ. To ensure that a S_{n-1} has n points exactly with the curve in common, necessitates that the degree of the R_i is $= n$.

It follows from the definition that no S_k can have more than $k + 1$ points in common with the curve (unless the curve is wholly contained in the S_k), as otherwise through this S_k and $n - k$ additional points belonging to the curve a S_{n-1} might be constructed, having more than n points in common with the curve.

The curve is uniquely determined by any $n + 3$ of its points; and between any $n + 4$ of its points a certain condition is fulfilled (from which for $n = 2$ the well-known Chasles and Pascal theorems for conics are easily deducible). To construct this condition and verify this proposition, let us return to the article entitled "Metrical Relations," &c., of NATURE, August 8. There it was pointed out that a point and a S_{n-1} may have a peculiar situation in regard to a pyramid of n manifoldness, by virtue of which to each point of the S_n corresponds one S_{n-1}, and *vice versâ*. It is not difficult to verify that when the coordinates of the point in regard to the pyramid are

$$a_1 \; . \; . \; . \; a_{n+1},$$

then the coordinates x_i of the points of the S_{n-1} satisfy the condition

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} \; . \; . \; . + \frac{x_{n+1}}{a_{n+1}} = 0.$$

If point and S_{n-1} have that relation to a pyramid, then they may be called pole and polar to it. It will be remembered that the construction of pole to polar, and *vice versâ*, is a purely projective one, by means of cuts of plane spaces, &c. The relation of $n + 4$ points of the curve to each other is now, that the polars of any three with regard to the pyramid of the other $n + 1$ have a S_{n-2} in common.

Indeed, let A_1 . . . A_{n+1} be $n + 1$ points of the curve, and P any of its other points, also

$$(\Sigma\chi_i) \,.\, P = \chi_1 A_1 + \; . \; . \; . + \chi_{n+1}A_{n+1} \text{ and } \chi_i = R_i(\lambda, \mu).$$

Then, A_1 being a point of the curve, R_2 . . . R_{n+1} must have a common zero point; and the same is true for $R_1 R_3$ . . . R_{n+1}; $R_1 R_2, R_4$ . . . R_{n+1}, &c. It is therefore easily seen that the coordinates of P may be put into the form

$$\chi_i = \frac{1}{a_i\lambda + b_i\mu}, \text{ where } a_i \text{ and } b_i \text{ are constants.}$$

The polars to P form, therefore, a pencil; that is, they have a S_{n-} in common.

If the points of the curve are projected from any one of its points into a S_{n-1}, they form a curve of the class considered in that space (as can be verified from the representation of the coordinates by parameters). For $n = 1$ the curve becomes a straight line, whose points form a homographic range with that (auxiliary) line, whose points are the representatives of the parameters (λ, μ). It follows, therefore: four points of the curve form with any group of $n - 1$ curve-points $4 S_{n-1}$ of constant cross-ratio.

If the curve degenerates, it degenerates always into straight lines or curves of the same class. This follows almost immediately from the definition. It is also obvious, that each degeneration implies the occurrence of at least one double-point. A twisted cubic may, for instance, degenerate into a conic and a straight line, that has with it a point in common (but is not situated in the same plane), or into three straight lines, of which one has one point in common with each of the other two.

In each point of the curve there is one straight line, that has two coinciding points in common with the curve, and one plane, that has three points of intersection which all coincide, &c. They may be called tangent lines, planes, &c., of the curve. Cut the curve by a S_{n-1}. If the n points of intersection are distinct, draw the n tangent S_{n-1} through them; and if only $n - 2$ are distinct, and 2 coincide, draw the $n - 2$ tangent S_{n-1}, and the one tangent S_{n-2}; and so on.

The point of intersection of these plane spaces may be called the *pole* of the original S_{n-1} to the curve; and this one, the *polar* of that point. The polar of any point of the polar passes the pole. Let the pyramid of reference be chosen so that the equation of the curve is

$$\chi_1 = \lambda^n \quad \chi_2 = \lambda^{n-1}\mu \; . \; . \; . \quad \chi_{n+1} = \mu^n.$$

The S_{n-1} may satisfy the equation

$$p_1\chi_1 + \; . \; . \; . + p_{n+1}\chi_{n+1} = 0.$$

The n points of intersection are then given by

$$p_1\lambda^n + \; . \; . \; . + p_{n+1}\mu^n = 0.$$

Their roots may be

$$\lambda/\mu = \alpha_1, \; \alpha_2 \; . \; . \; . \; \alpha_n.$$

Through $\chi_1 = \alpha^n \; \chi_2 = \alpha^{n-1}$ . . . the tangent S_{n-1} (whose coordinates may be ξ_i) $a_1\xi_1 + \; . \; . \; . + a_{n+1}\xi_{n-1} = 0$ will be such that

$$a_1 = 1 \quad a_2 = n \,.\, \beta \quad a_3 = (n)_2\beta^2 \; . \; . \; . \; a_{n+1} = \beta^n,$$

where β is a parameter, whose value is found $= -\alpha$. The point of intersection of the n S_{n-1}, whose equations are

$$\xi_1 - n \,.\, \alpha_i \quad \xi_2 + (n)_2 \,.\, \alpha_i^2 \quad \xi_3 - \,.\, \pm\alpha_i^n\xi_{n+1} = 0$$

is obviously

$$\xi_{+n1} = p_1 \quad \xi_n = -\frac{p_2}{n}$$

$$\xi_{n-1} = \frac{p_3}{(n)_2}, \; \text{\&c.}$$

(on account of the equation satisfied by the α).

If ξ_i is any point, and χ_i any point on its polar, the equation exists

$$\xi_{n+1}\chi_1 - n\xi_n\chi_2 + (n)_2\xi_{n-1}\chi_3 - \; . \; . \; . = 0,$$

which is symmetrical, and therefore proves the proposition.

The polar to a line joining two points is the cut of their polars; and so generally. It is therefore possible to speak of the polar, or pole, of any plane space, in regard to the curve. The two are united only when the two sets of coordinates are equal, that is, when they satisfy a condition of the second degree. Pole and polar cut a straight line in involution, as immediately follows from the symmetry of the equation connecting them. The double points of the involution are the points in which the straight line cuts that surface of the second order.

Much more could be said concerning this class of curves, the properties of which are so much like those of the conics; but I hope that what has already been mentioned will be found sufficient to interest mathematicians in their existence.

London, September 6. EMANUEL LASKER.

The Freezing Point of Silver.

THE subject of high temperature thermometry has recently attracted considerable attention, and on account of the ease with which silver can be obtained in a pure state, coupled with its great thermal conductivity, the freezing point of this metal has been suggested as a standard temperature. We therefore wish to call attention to an error into which we believe M. le Chatelier has fallen with regard to this constant. In the *Zeitschrift für Physikalische Chemie*, Band viii. p. 186, he says that the melting point of silver can be lowered by as much as 30° through the absorption of hydrogen; again, in the *Comptes rendus* for August 12, 1895, he states that the melting point of this metal is lowered by a reducing atmosphere. He therefore recommends that when the melting point of silver is used as a fixed point in calibrating pyrometers, the experiment should be performed in an *oxidising* atmosphere. This conclusion is contradicted by Prof. Callendar's experiments and by our own, for in the *Phil. Mag.*, vol. xxxiii. p. 220, Callendar shows that the freezing point of silver is lowered and rendered irregular by an oxidising atmosphere; and our own results confirm this