THE FOUNDATIONS OF DYNAMICS.[1]

THIS posthumous volume of Hertz's works, edited by Prof. Lenard, with a preface by von Helmholtz, has a doubly melancholy interest. It is the last work of Hertz upon which he was engaged until a few days before his death, and it contains a preface which is almost the last work of von Helmholtz. The pupil died shortly before his master, and by the departure of such a pupil and of such a master, science, and with science mankind, have lost many prospects of advances in the near future.

In his preface, von Helmholtz pays a touching tribute to the genius of his favourite pupil, from whom he hoped most, and who had drunk most deeply of his master's thoughts. In 1878 their intimacy began. At that time difficulties connected with various electrical theories of action at a distance were occupying his thoughts, and he offered a prize for the best essay on induction in non-inductively wound coils. Weber's theory would have involved an inertia of the electric current distinct from the magnetic inertia. The question is still interesting in connection with discharges between two charged conductors, one of which completely surrounds the other when a dielectric between them is suddenly made conductive. There is then no magnetic force. Is there *no* inertia? Can a medium become *suddenly* conducting? Is a conducting medium homogeneous? Is there inertia of ionic charges which represent the non-homogeneity of the medium? These questions still require answering; but in the seventies, in Germany, Maxwell's idea of magnetic force accompanying displacement currents was not generally received, and Helmholtz's question as to the induction in non-inductively wound coils really had reference to these displacements. Hertz won the prize by showing that at most only 1/20th or 1/30th of the extra-current could be due to electric inertia. By subsequent experiments on the possible effect of centrifugal force on the current in rapidly rotating plates, he reduced this estimate to a very much smaller value. Mr. Larmor has suggested that any centrifugal force may be balanced by a tension in the length of the current, much in the same way that the tension of a running rope will balance centrifugal force in the curves round which it may be running. In every way the subject deserves further investigation, for it is intimately connected with the most fundamental questions as to the nature of electricity and its connection with matter.

The next thing to which Hertz devoted himself was a prize problem proposed, at von Helmholtz's suggestion, by the Berlin Academy. The problem was to investigate Maxwell's postulate that changing electric displacement was an electric current. This was the bud from which Hertz's great work sprang. Of it von Helmholtz says: "It is a pity we do not possess more such histories of the inner psychological development of knowledge. Its author deserves our sincerest thanks for letting us see so deeply into the inmost working of his thoughts, and for recording even his temporary mistakes. By this work Hertz has settled for ever the question as to electromagnetic actions being propagated by a medium, and the only outstanding question of the kind is as to gravitation, which we do not yet know how to logically explain as other than a pure action at a distance." It thus appears that von Helmholtz to the last was unconvinced as to the probability of any hypothesis like Le Sage's or Osborne Reynolds's. He seems, on the other hand, to have been satisfied with the possibility of chemical actions being explained either by electromagnetic actions or by actions not at a distance. This latter term, of course, requires explanation as to what "at a distance" means. Any actions other than those of absolutely rigid bodies, such, for instance, as the fairly well-established forces of attraction of gaseous molecules for one another, and some of which can hardly be explained either by electricity, magnetism, or gravitation, seem to be actions at a distance that require explanation just as much as gravitation.

Following this short history of the work of his pupil which, coming from such a master, must have a permanent interest to all, von Helmholtz gives a *résumé* of the last work of Hertz. In it there is attempted a continuously elaborated presentation of a complete and self-dependent system of mechanics, in which each particular application of this science is deduced from a single fundamental law which can of course be itself only assumed as a plausible hypothesis. In order to explain how this is required, von Helmholtz gives a short history of the development of the science of mechanics. The first developments arose from the study of the equilibrium and motion of solid bodies in direct contacts with one another, such as the simple machines, the lever, the inclined plane, the pulley. The law of virtual velocities gives the most fundamental general solution of all such problems. Galileo subsequently developed the knowledge of inertia and of moving force as an accelerating agent. It was, however, conceived by him as a succession of blows. Newton was the first who arrived at the notion of force acting at a distance, and its more accurate determination by the principle of action and reaction. It is well known how strenuously he and his contemporaries resisted this idea of pure action at a distance. From this men developed the methods of treating all problems of conservation forces with constant connections whose most general solution is given by D'Alembert's principle. All the general principles of dynamics have been developed from Newton's hypothesis of permanent forces between material points and permanent connections between them. It was subsequently found that these laws held even when these foundations could not be proved, and it was thence deduced that all the laws of nature agreed with certain general characteristics of Newton's conservative forces of attraction, although it was not found possible to deduce all these generalisations from one common fundamental principle. Hertz has devoted himself to discovering such a fundamental principle for mechanics, from which all the laws of mechanics hitherto known as universally valid can be deduced; and he has carried out this with great acuteness, and by means of a very remarkable presentation of a peculiarly general kinematic conception. In working it out, he returns to the oldest mechanical theories, and supposes all actions to be by means of rigid connections. Of course he has to assume that there are innumerable imperceptible masses and invisible motions of these, in order to explain the apparent actions upon one another of bodies that are not in immediate contact. Though he has not given examples of how this may be the case, he evidently builds his expectation of being thus able to explain natural actions upon the existence of cyclical systems, rollers, &c., with invisible motions. The justification of such an assumption can only be obtained by its success. Von Helmholtz concludes this interesting preface by remarking how English physicists have so often based their work on dynamical and geometrical suppositions, as for example Lord Kelvin and his vortex atoms, Maxwell and his cells with rotating contents. These physicists, he says, "have clearly been more satisfactorily helped by such illustrations, than by the mere most general representations of the facts and their laws as given by the system of physical differential equations. I must confess that I have restricted myself to this latter method of investigation, and have felt most confidence therein; and indeed I might not have arrived at any important results by the methods which eminent physicists such as the three mentioned have employed."

Although so far it seems as if there were very little to choose between the old methods of supposing that natural actions can be explained by conservative forces between molecules and by systems of rigid connections, Hertz in his introduction shows that he is dissatisfied with the hypotheses, of these forces as entities, while von Helmholtz, by his silence, seems to hold the view that the old method was good enough for him. Hertz's method has, however, the advantage of turning our attention to something definite to be investigated and invented, namely, the structure of these rigid connections. It is apparently very closely related to Osborne Reynolds's and "Waterdale's," suggestions as to the structure of the ether, namely, that it consists of perfectly rigid particles in almost complete juxtaposition which, whether by their smoothness or by their rolling upon one another, waste no energy in internal heat motions.

In his own preface, Hertz says that he has culled many things from many minds, nothing particular in his work is new; what he presents as new is the arrangement and collocation of the whole, and the logical, or rather philosophical, aspect thereby attained.

To these prefaces there follows a long introduction, in which Hertz reviews and criticises the present foundations of dynamics. The great road by which this domain is now entered is one that was laid by Archimedes, Galileo, Newton, and Lagrange. It is founded on our notions of space, time, force, and mass. Force is introduced prior to motion, as the independent cause thereof. Galileo's notion of inertia only involved space, time, and mass.

[1] 'The Principles of Dynamics developed on new lines:—Hertz's Collected Works," vol. iii. Pp. 310. (Leipzig: Barth, 1894.)

Newton first introduced all four notions. To this D'Alembert's principle gave the analytical method of treating generally connected systems. Beyond it all is deduction. Here Hertz introduces a discussion as to the so-called forces of inertia. From his discussing the case of a solid subject to centripetal acceleration by means of a string, the question is much more intricate than if he had taken the case of a body falling freely under gravity, where the force is applied directly by the earth to each point of the body, and not, as in the case of the string, distributed to each part by stresses in the solid. Hertz seems to consider that there is some outstanding confusion in applying the principle of equality of action and reaction, and appears to hold that by this principle the action on the body requires some reaction *in the body* whose acceleration is the effect of the force. He does not seem fully to appreciate that action and reaction are always on *different* bodies. From his consideration of this, and from a general review of our conception of force, he concludes that there is something mysterious about it, that its nature is a problem in physics, like the nature of electricity. We have a quite distinct conception of velocity: why not of force? He concludes that the mystery is not due to our not having enough ideas to associate with the word, but to our trying to put too much into it. These mysteries, however, do not invalidate in any way the deductions that have been made; they only require us to seek out a new foundation for our dynamics. He goes on to criticise this method of filling nature with forces of which, being ultimately between molecules, we can have no direct experience. A piece of iron on a table is acted on by gravitation, cohesion, repulsion, magnetic, electromagnetic, electric, and chemical forces. Some of these would drag it to pieces if unbalanced to a nicety by others. Is this a sound view of nature? Can we not get some more attractive one?

A second view may be elaborated by making our fundamental quantities, space, time, mass, and energy. There is no book in which this view of nature is fully and consistently worked out, at least none that Hertz was acquainted with. He sketches how it might proceed. Besides the postulate of the conservation of energy we require some definition of potential energy and experimental relations connecting it with space, and in addition we have a choice of relations with kinetic energy, of which Hertz suggests the choice of the integral form of Hamilton's principle known as that of least action. This is, no doubt, a recondite idea to use as a fundamental postulate, but it only implicitly involves the idea of force, which then comes in merely as a definition. To this method, which certainly has several great advantages, Hertz makes a number of objections. In the first place he objects that it requires the equations of connection to be integral equations, and we know such actions as pure rolling of one hard body on another cannot be so expressed. We must, in order to specify the subsequent motion, know the rate of rotation round the normal axis through the point of contact, and this cannot be specified except in terms of differentials. To such motions we cannot apply the proposed principle of least action, and yet we can hardly dispute that such rolling is possible in nature. If we treat it as the limits of frictional sliding, we introduce the whole of the difficulties of force, or of the irregular heat actions which have not yet been fully made amenable to accurate dynamical treatment. Again, difficulties arise as to the foundation of this method. There is great difficulty in specifying energy itself. How can it be satisfactorily measured without returning to the first method, and introducing the idea of force? Some have conceived of energy as a sort of substance; but when we try to form concrete conceptions of what is occurring, we get involved in perplexities. The very existence of two forms of energy is a very serious difficulty. Again, it is doubtful whether it can be sound to consider the integral of least action as a *fundamental* principle. It makes the present depend on the future. It sets the problem to nature to make a certain integral the minimum.

A good many of these objections could be got over by making all energy kinetic, which is what Hertz himself practically assumes in his own method.

This third method begins by assuming only three fundamental quantities, time, space, and mass, and puts aside as non-fundamental, force and energy. In order to explain how nature works, we already do postulate invisible underlying structures in nature. We postulate these in the atoms and molecules of matter. Hertz sees in all actions the working of an underlying structure whose masses and motions are producing the effects on matter that we perceive, and what we call force and energy are due to the actions of these invisible structures, which he implicitly identifies with the ether.

We must, however, assume certain connections between the three quantities, time, space, and mass. Between time and mass there is no direct connection. Space and mass, Hertz considers, are connected by the existence of a given mass at each point of space. He cannot mean here to assume a complete plenum, which would make serious difficulties in the way of the working of what he subsequently assumes to be a structure of rigid bodies; he must include a vanishingly small density at some points, though perhaps he may have had in view the filling of the interstices between his rigid bodies with a fluid. Any way, he goes on to say that some connection is required between all three quantities, and for this purpose he postulates his great fundamental single law of motion, which he considers is an extension to systems of Newton's first law of motion for a single body; it is that a system, which is unconnected with any others, moves with constant swiftness along one of its straightest paths. "Systema omne liberum perseverare in statu quo quiescendi vel movendi uniformiter in directissimam." In order to understand what Hertz here means by the path of a system, and by its being straight or curved, requires further explanation; but from this principle, which is capable of analytical representation, and from the assumption that the connections of a system are all rigid, he deduces all the fundamental principles, conservation of areas, momentum, energy, least action, &c. In considering the motion of any part of a system, we find that we may conveniently introduce certain actions of the other parts of the system upon it which are measured by forces, which thus come in as mere definitions. He does not seem to investigate anywhere the question as to the danger of his rigid connections becoming tangled. Analytically a postulate that the points of two different bodies that act on one another are in contact is easily expressed, but it does not follow that when we come to invent actual rigid connections to produce the observed effects, they will do so for any length of time without jamming. It is a seductive theory that gravitation or electrical actions may be due to vortex filaments ending on atoms; but the tangling of the filaments is a very serious difficulty that has not been satisfactorily got over. Hertz does not seem to feel this as a serious difficulty, but he does notice an obvious objection that is sure to be raised, namely, that rigidity in itself postulates forces. To this he replies that rigidity in itself is merely a matter of definition and of fact. How is our view of the fact that two points are at a constant distance apart, improved by saying that there is a force between them? As, however, real bodies are only imperfectly rigid, Hertz concedes that it may be that when we learn more about these invisible connections, they may turn out not to be absolutely rigid. It is a matter for further investigation. This very same view might have been urged, and has been urged already with reference to actions like gravity. The *law* of gravity can be perfectly well described without any reference to the notion of force. We may say, every element of matter moves towards every other element in the universe with an acceleration inversely proportional to the square of their distances apart. We can describe the law kinetically, just as Hertz proposes to describe the law of motion of parts of a rigid body. There is no *necessity*, however convenient it may be, to introduce the notion of force; the other bodies in the universe are a sufficient cause for motion of each, without postulating an entity, force. The principal reason for introducing this notion was to account for a body acting where it was not; force was invented to get over this; the body produced force, and this force existed where the body did not, and there acted on other bodies. This whole difficulty seems, however, to be partly due to want of distinct ideas connected with the question of where a body *is*. We are so accustomed to consider a body as having a definite boundary, that we think there is a definite boundary in reality. All we know of the atoms and molecules, however, would lead us to conclude that round the centre of each there is a very complexly structured region which may or may not change abruptly in structure, but which often extends to considerable distances from the atom, so that it is practically impossible to state absolutely where the atom ends and where the empty space begins. With this view of matter there is no serious reason why we may not rightly consider each atom as existing everywhere that it acts, that is, throughout the whole of space, for its action in causing gravitional accelerations exists, so far as we know, throughout space. A view of this kind entirely gets over any difficulty of a body acting where it is not; for all bodies *are* everywhere, and if we consider matter to be the cause of motion of other matter, there seems no very imperious necessity for imagining another cause which we call force.

There are two assumptions that Hertz makes which he considers can only be proved by their success. One is that all the connections in nature can be represented by linear differential equations. There are plenty of cases imaginable in which this would not be true, as, for example, connections depending on the curvature of the path. The other assumption is that forces can be represented by force functions. This, again, may not be a complete representation of nature.

Following this introduction comes the book itself, which is divided into two parts. The first part is purely kinematical, the second deals with the deductions from Hertz's fundamental postulate of motion in the straightest possible path.

The first part begins by explaining what is meant by the path of a system of points. To get at this we calculate the mean square of the displacements of a system of points when they are displaced: the square root of this, Hertz calls the displacement of the system of points. If there is a mass at each point, then the displacement of the system is the square root of the mean squares of the displacements of the points, each multiplied by the mass at it. Thus, if s be the displacement of the system, and $s_1 s_2$, &c., the displacements of each point of masses $m_1 m_2$, &c. Then

$$(m_1 + m_2 + \ldots\ldots)s^2 = m_1 s_1^2 + m_2 s_2^2 + \ldots$$

By taking s_1, s_2, &c., as the displacements in the element of time, we evidently get a similar expression for the velocity of the system, and for its acceleration. The mean square of the velocity of the parts of a system is well known in connection with the principle of least action. Further than this, however, Hertz defines the angle between two displacements. This is defined by the equation

$$(m_1 + m_2 + \ldots\ldots)ss' \cos \epsilon = (m_1 s_1 s_1' \cos \alpha_1 + m_2 s_2 s_2' \cos \alpha_2 + \ldots\ldots)$$

s and s' being the two displacements of the system as calculated above, and $s_1 s_1'$, &c., the two displacements of each point and α_1, α_2, the angles between these latter, then ϵ is the angle between s and s'. Hertz remarks that these can all be very interestingly expressed in terms of space of multiple dimensions, in which analytical diagrams are supposed to be drawn. This, however, represents the real by the unattainable. There follow, then, several chapters expressing these displacements in terms of various systems of coordinates, and discussions as to the conditions that the connections of a system should fulfil in order that they may be represented by equations not involving differentials. The curvature of the path is here studied. It is defined as $c = \frac{d\epsilon}{ds}$, and from this it follows that, representing $\frac{d^2x}{ds^2}$ by x'', &c.

$$(m_1 + m_2 + \ldots\ldots)c^2 = \Sigma_1{}^1(m_1 x_1''^2 + y_1''^2 + z_1''^2).$$

The problem then of making the path of the system straightest, is to make c a minimum consistently with the connections of the system. Now, in accordance with his assumption that the connections of the system are linear differential equations of the form

$$\Sigma_1{}^1 P_1 x'_1 = 0,$$

whose differentiation gives

$$\Sigma_1{}^1 P_1 x'_1{}' + \Sigma^1 \Sigma_1{}^1 . \frac{dP_1}{dx_2} x'_1 x'_2 = 0,$$

we are to determine the minimum value of

$$c^2 = \Sigma_1{}^1 \frac{m_1}{m} . x''_1{}^2,$$

when

$$m = m_1 + m_2 + \ldots\ldots$$

In determining the variations of these, we must recollect that the positions and direction of displacement, *i.e.* the first differentials of the system, are supposed given, and that it is *only* the second differentials that can be varied in order to make c a minimum. Calling, then, a system of indeterminate coefficient λ, μ, &c., corresponding to the equations of condition, we evidently get a system of equations of the form

$$\frac{m_1}{m} x''_1 + \Sigma_1{}^1 P_1 \lambda = 0,$$

which are sufficient to determine the second differentials required.

From this form of result one can see how the ordinary equations of motion are derivable from the conception of the straightest path, and how, when dealing with part of a system, these indeterminate coefficients introduce what are equivalent to forces. This method of deducing the equations of motion lends itself particularly well to the deduction of the principles of least action, and the other general methods in dynamics. So far, he deals with free systems subject only to internal constraints. It is where he investigates how to deal with parts of systems that he requires to consider the nature of the constraints joining one part to another. For this purpose he defines two systems as coupled when coordinates can be so chosen that one or more of them are the same for both systems. Force is then defined as the action one system has on another. Now, when a coordinate is the same for two systems, one of the equations of condition is $p = p'$, p and p' being coordinates of the coupled systems, and for this equation the coefficient P becomes the same in the two systems, being unity for each, so that the equations of motion involve the indeterminate coefficient λ corresponding to this equation equally with reference to each system. It is thus that the equality of action and reaction appears, being thus bound up with the constant equality of the common coordinate. This seems to be where the assumption that the connections are rigid is introduced. When rigid bodies act upon one another by non-slipping contact, certainly the coordinates of the point of contact are common to the two systems. It is also quite evident that if we assume rigid bodies acting upon one another by contact only, we can have no potential energy, and all *necessity* for talking about the forces disappears. In Hertz's system there are no forces like Newton's acting between bodies which have no common coordinate, like the earth and the sun. We would have to invent connections to explain the motion before we could be certain that action and reaction are equal in this case.

The proof of the principle of virtual velocities by substituting for the forces between parts of a system a number of pulleys which produce the same effects, is quite analogous to Hertz's supposition that the actual connections are by rigid bodies. It is not, however, liable to the objection that the connections may become tangled, for it is only applied to the case of infinitesimal virtual displacements, while Hertz postulates the possibility of his connections existing as the real ones for all time, and throughout all finite displacements of the system.

The work considers many other matters, and shows how all the general methods in dynamics are deducible from his fundamental postulate of the straightest path. It includes discussions on how best to deal with systems whose connections do not involve differentials, how to treat cyclical coordinates, and many other matters. It is most philosophical and condensed, and gives one of the most—if not the most—philosophical presentations of dynamics that has been published. It is worthy of its author: what more can be said?

G. F. FITZGERALD.

PSEUDO-SATELLITES OF JUPITER IN THE SEVENTEENTH CENTURY.

IN the New York *Nation* for January 11, 1894, Dr. D. C. Gilman, President of the Johns Hopkins University, called attention to an interesting letter from John Winthrop, jun., to Sir Robert Moray, concerning the satellites of Jupiter. In this letter, which was written from Hartford, Connecticut, on January 27, 166⅘, Winthrop described an observation of Jupiter which he had made on the night of the previous 6th of August, when he had very distinctly seen five satellites about that planet. He was naturally "not with out some consideration whether that fifth might not be some fixt star with which Jupiter might at that tyme be in neare conjunction," and expressed the wish that more frequent observations might be made upon that planet with a view to ascertaining whether it is not impossible to discern a fixed star, when it is so near to the planet as to appear "within the periphery of that single *intuitus* by a tube which taketh in the body of Jupiter," and if