to the *Länderkunde*. Although fully recognizing the difficulty of having lectures in all the above-named subjects especially appropriated to the needs of geography, the Council suggest that *privat-docents* might supply the new want. But if this is found to be impossible, they advise that the students who wish to take either geography or anthropology as their specialty should be left to select in the above-named group of sciences those subjects which would best suit them. Students might thus take any one of the three chief directions opened to the geographer—namely, that of the geologist-geographer, the biologist-geographer, or the anthropologist-geographer.

THE MULTIPLICATION AND DIVISION OF CONCRETE QUANTITIES.[1]

I HAVE recently been laying stress on the fact that the fundamental equations of mechanics and physics express relations among *quantities*, and are independent of the mode of measurement of such quantities; much as one may say that two lengths are equal without inquiring whether they are going to be measured in feet or metres; and indeed, even though one may be measured in feet and the other in metres. Such a case is, of course, very simple, but in following out the idea, and applying it to other equations, we are led to the consideration of products and quotients of concrete quantities, and it is evident that there should be some general method of interpreting such products and quotients in a reasonable and simple manner. To indicate such a method is the object of the present paper.

For example, I want to justify the following definition, and its consequences: Average velocity is proportional to the distance travelled and inversely proportional to the time taken, and is measured by the distance divided by the time, or, in symbols, $v = s \div t$. As a consequence of this, the distance travelled is equal to the average velocity multiplied by the time, or $s = vt$. The following examples will serve to illustrate what I mean:—

(i.) If a man walks 16 miles in 4 hours, his average speed is $\frac{16 \text{ miles}}{4 \text{ hours}} = 4 \times \frac{1 \text{ mile}}{1 \text{ hour}} = 4$ miles an hour, the symbol $\frac{1 \text{ mile}}{1 \text{ hour}}$ denoting a speed of a mile an hour, in accordance with the definition.

Similarly, $\frac{1 \text{ foot}}{1 \text{ second}}$, or shortly, $\frac{\text{ft.}}{\text{sec.}}$, denotes a velocity of a foot per second. The convenience of this notation is that it enables us to represent velocities algebraically, and to change from one mode of measurement to another without destroying the *equation*.

Thus $\frac{16 \text{ miles}}{4 \text{ hours}} = \frac{4 \text{ miles}}{1 \text{ hour}} = \frac{4 \times 1760 \times 3 \text{ feet}}{60 \times 60 \text{ seconds.}} = 5{\cdot}9 \frac{\text{ft.}}{\text{sec.}}$ $= 5{\cdot}9$ feet per second.

(ii.) The distance travelled in 40 minutes by a person walking at the rate of $4\frac{1}{2}$ miles an hour $= \frac{4\frac{1}{2} \text{ miles}}{1 \text{ hour}} \times 40$ minutes $= \frac{4\frac{1}{2} \text{ miles}}{3} \times 2 = 3$ miles.

Such concrete equations are used by a considerable number of people, I believe, but I have not seen any attempt at a general method of interpreting the concrete products and quotients involved.

Now, I think I cannot do better by way of clearing the ground before us than quote what Prof. Chrystal says in his "Algebra" about multiplication and division. He begins by saying that multiplication originally signified mere abbreviation of addition; and then (on p. 12) he says:—

"Even in arithmetic the operation of multiplication is extended to cases which cannot by any stretch of language be brought under the original definition, and it becomes important to inquire what is common to the different operations thus comprehended under one symbol. The answer to this question, which has at different times greatly perplexed inquirers into the first principles of algebra, is simply that what is common is the formal laws of operation [the associative, commutative, and distributive laws]. These alone define the fundamental operations of addition, multiplication, and division, and anything further that appears in any particular case is merely a matter of some interpretation, arithmetical or other, that is given to a symbolical result, demonstrably in accordance with the laws of symbolical operation."

"*Division*, for the purposes of algebra, is best defined as the inverse operation to multiplication."

I will begin by considering instances, and then go on to the general case.

A product of a number and a concrete quantity presents no difficulty. All that is necessary is to define that the order of stating the product shall not alter its meaning—that is, that the *commutative* law shall hold—that,

$$e.g., \ 2 \times 1 \text{ foot} = 1 \text{ foot} \times 2 = 2 \text{ feet}.$$

The *distributive* law is satisfied; thus,

$$2 \text{ feet} + 3 \text{ feet} = (2 + 3) \text{ feet} = 5 \text{ feet}.$$

In interpreting the meaning of the product of two concrete quantities, we have to be careful that in the interpretation nothing shall violate the laws of *numerical* multiplication; *i.e.* if any numerical factors occur, they must be able to be multiplied in the ordinary way, and placed before the final concrete product, which must, of course, represent something which varies directly with both quantities.

Thus 4 feet × 2 yards must be equal to 8 × 1 foot × 1 yard.

Now a *rectangle*, whose sides are 4 feet and 2 yards, is eight times the rectangle whose sides are 1 foot and 1 yard, so that, if we define the product of two lengths as representing a rectangle whose sides are these lengths respectively, we are not violating any multiplication law as regards the numerical multipliers; and we can compare one such rectangle with any other whose sides are of different lengths, by ordinary multiplication and division among such numbers as arise, and by interpretation of the concrete products in accordance with the definition.

$$\begin{aligned} \text{Thus, } 4 \text{ feet} \times 2 \text{ yards} &= 8 \times 1 \text{ foot} \times 1 \text{ yard}, \\ &= 24 \times 1 \text{ foot} \times 1 \text{ foot}, \\ &= 24 \text{ square feet}, \\ &= 24 \times 12 \text{ inches} \times 12 \text{ inches}, \\ &= 3456 \text{ square inches}, \\ &\text{\&c.} \end{aligned}$$

Here we have applied the commutative law so as to bring the numerical factors together for multiplication, and have interpreted the remaining concrete products in accordance with the definition.

The general result is that $ab = \alpha\beta \, . \, a'b'$, if $a = \alpha a'$, and $b = \beta b'$, *i.e.* a rectangle whose sides are a, b is $\alpha\beta$ times a rectangle with sides a', b', if $a = \alpha a'$, and $b = \beta b'$.

From this example I think we can see that a concrete product may properly be used to represent any quantity that varies directly as the several concrete factors, and that, being so represented, it may, by use of the ordinary rules of multiplication, be compared with any other concrete product of the same kind; that is to say, that, *generally*, $ab = \alpha\beta \, . \, a'b'$, if $a = \alpha a'$, and $b = \beta b'$, where α, β are numerical factors, and a, a' are different amounts of one kind of quantity, and b, b' of another kind.

Similarly, a concrete quotient may be used to represent a quantity which varies directly as the concrete numerator and inversely as the concrete denominator, and may, by the ordinary rules of multiplication and division, be compared with any other quantity of the same kind.

Indeed, I may go further and assert that a concrete product or quotient (the latter including the former) MUST, if it is to have any meaning at all, represent a quantity varying directly as the concrete factors in the numerator and inversely as those in the denominator, and that the general use of such representation is for comparison of the complex quantity with a standard of the same kind. Or, generally, we may say it should be used, whenever we wish, in our work, to give as full and explicit a representation to the complex quantity as possible.

The operation of multiplying [and dividing] concretes may be separated into two parts: the *formation* of the products, and the *simplification* of them; and this latter process may be again considered in two parts: the simplification of the numerical factors, *i.e.* ordinary multiplication and division, and the simplification of the concrete factors, *i.e.* cancelling where possible, and, finally, interpretation.

[1] Paper read at the General Meeting of the Association for the Improvement of Geometrical Teaching, on January 14, 1888, by A. Lodge, Cooper's Hill, Staines.

The first part of the multiplication is the *representation* of a complex quantity which is proportional to the several factors in the numerator, and inversely proportional to those in the denominator; the second part is the comparison between the particular complex quantity and a standard of the same kind. The representation may be temporary, *i.e.* adopted for the solution of a particular problem; or it may be permanent, *i.e.* adopted throughout a whole subject.

Thus, if a, b are two lengths, the product ab is always used to represent a *rectangle* whose sides are a, b respectively; though we *might* have agreed to use it as a representation of a parallelogram with sides a, b containing an angle of (say) 60°; and of course we might find a number of things which in some particular problem might be represented by ab, but all such quantities must agree in this property, viz. that in the problem in question they shall vary jointly as a and b.

Our right to cancel among concretes may be established once for all in some such way as the following:—

Let $a = \alpha a'$, $b = \beta b'$, and therefore $ab = \alpha\beta \cdot a'b'$, as before. Now, if we proceed to deduce a formally from the equation $ab = \alpha\beta \cdot a'b'$, we shall get $a = \frac{\alpha\beta \cdot a'b'}{b}$, which reduces down to its known value $\alpha a'$ if we allow b in the denominator to cancel against its equivalent $\beta b'$ in the numerator. (This cancelling is really an application of the law of association to the quotients.)

By such methods as this we can establish once for all our right to apply the formal laws of multiplication and division to concrete products and quotients, when such concrete products and quotients represent quantities varying directly as the concrete numerator and inversely as the concrete denominator; though, indeed, for that matter a very little practice in the use of such concrete representations renders one's perception of that right almost intuitive. In fact, in all cases a student would very soon perceive that the standards involved in the various equations might be treated exactly like numbers, and he would also learn from the resulting expressions $\left(e.g.\ \frac{\text{foot}}{\text{sec.}}, \frac{\text{foot}}{(\text{sec.})^2}, \&c.\right)$ to appreciate the meaning of the *dimensions* of quantities with a thoroughness unattainable in any other way.

All questions dealing with mixed standards, or change of standards, present no difficulty when this method is adopted.

Here is a good example of the concrete method. Two ton-masses placed a yard apart attract each other with a force equal to the weight of one-eighth of a grain. Calculate the mass of the earth in tons.

Solution. $\frac{\text{earth} \times \frac{1}{8}\text{ grain}}{(4000\text{ miles})^2} = \frac{1\text{ ton} \times 1\text{ ton}}{(1\text{ yard})^2}$

$$\therefore \text{mass of earth} = \frac{1\text{ ton}}{\frac{1}{8}\text{ grain}} \times \left(\frac{4000\text{ miles}}{1\text{ yard}}\right)^2 \text{tons}$$
$$= \&c.$$

It is most important that the student should be taught to notice that physical equations can only be among quantities of the same kind, or that, if there are quantities of different kinds in the equation, then the equation is really made up of two or more independent equations which must be separately satisfied, each of these being only among quantities of the same kind. So we may consider generally that, in any equation, all the terms must represent quantities of the same kind.

But I want to call attention to the fact that merely the dimensions of a quantity do not always fix the kind of quantity. For example, the moment of a force is of the dimensions of work, and yet it is not work, and cannot exist as a term in an equation involving *work* terms. Again, the circular measure of an angle is not a pure number, though it is of zero dimensions as a pure number is; and that it is not a pure number is evident physically, for a moment of a force × an angle = work.

Now these are special cases of certain general laws as to direction which hold among the terms of an equation involving directed quantities, *but in which the symbols themselves do not include the idea of direction* (for I wish to confine myself strictly to ordinary algebraical equations).

The laws are: firstly, if any term is independent of direction, every term must be also independent of direction, or involve ratios between *parallel* vectors, and so by cancelling direction become independent of it.

E.g. if a body is projected with velocity V at angle α with the horizon, it reaches its greatest height in the time $\frac{V \sin \alpha}{g}$.

Here both numerator and denominator are vertical vectors, and therefore the directions cancel as they ought.

Secondly, if any term involve only one vector, the other terms must also, after such simplification of directions as possible, involve *the same* vector only.

E.g. Horizontal range of projectile $= \frac{2V^2 \sin \alpha \cos \alpha}{g}$, where V sin α and g are vertical vectors, and V cos α is horizontal, so that the whole expression is a horizontal vector, as it should be.

Again, if any term involve a product (or ratio) between two vectors including any angle, every term must, after such cancelling and simplification of directions as possible, also involve a product (or ratio) between two vectors including the same angle.

The most frequent cases are those where a term consists of a product of parallel, or mutually perpendicular directed quantities, in which case every term must do the same.

It is not easy to see what law holds in cases where a greater number of directed quantities occur in each term, except in the simple case where one term consists of a product of a number of parallel vectors, in which case every term must do the same.

The general law is, I believe, that if any term consists in its simplest form of a product or quotient of certain vectors, which will form a kind of solid angle, then every term must also involve an exactly *similar* solid angle of vectors. However, I have not followed this out, as it does not seem likely to be a useful test in its general form.

The following are simple examples of some of the above laws:

$$\left.\begin{array}{l} b = a \cos C + c \cos A \\ a^2 = b^2 + c^2 - 2bc \cos A \end{array}\right\} \text{in a triangle;}$$
$$y = mx + c;$$
$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

This last example should be considered in connection with the ordinary geometrical proof, where it will be seen that each term on the right is a ratio between lines inclined to each other at the angle 90° − (A + B), just as the left-hand side is.

An angle is the ratio between the arc and radius of a circle, and if it multiplies a radius, changes it into an arc. Thus, if by applying a force P at the end of an arm a, a body is turned through a small angle θ, the work done is $Pa\theta$; *i.e.* the product of P into the arc through which it has been acting, which is a product of *parallel* vectors, as it must be besides having to be of right dimensions if it is to represent work. This expression is also the product of the moment of the force into the small angle turned through, so that, if we wish to connect the moment of a force with work, we must say:—

The moment = the work *per radian* which can be done,

$$\text{or simply, moment} = \frac{\text{work done}}{\text{angle turned through}}$$

Now I do not wish to insist that in dealing *practically* with mechanical problems it is necessary always to include the standards as well as the numerical multipliers in the equations, for it would be an intolerable nuisance to have to do so. In complicated cases, however, I think the student should test the dimensions of each term in his equation, so as to avoid gross mistakes. But it is in trying to *understand* the fundamental equations in any subject that it appears to me important to express particular examples of them as fully as possible.

For practical purposes any numerical equations we may desire may be deduced from the fundamental equations.

For example, the connection between the height (h) of an observer above the sea with the distance (d) of his horizon, is $d^2 = 2Rh$, where R is the radius of the earth; and we can deduce from this the numerical relation between the height in *feet*, and the distance of vision in *miles*. For if f be the number of feet in h, and m the number of miles in d, so that $h = f$ feet, and $d = m$ miles, the equation becomes

$$(m\text{ miles})^2 = 2R \times f\text{ feet},$$
$$= 8000\text{ miles} \times f\text{ feet};$$
$$\therefore f = m^2 \frac{(\text{miles})^2}{8000\text{ miles} \times 1\text{ foot}} = \frac{5280}{8000} m^2,$$
$$= \tfrac{2}{3} m^2 \text{ approximately};$$

i.e. the observer's height in feet = $\frac{2}{3}$ of the square of the distance of his view in miles.

This is a strictly numerical equation, deduced for practical purposes from the concrete equation $d^2 = 2Rh$.

It cannot, I think, be too clearly impressed on the student that, when any quantity is expressed by a number, that number is the *ratio* of the quantity to some standard of the same kind.

To take the preceding example, f is the number of feet in the height h.

$$i.e. \quad h = f \text{ feet},$$

$$\therefore f = \frac{h}{1 \text{ foot}} = \text{the } ratio \text{ of } h \text{ to 1 foot.}$$

$$\text{Similarly } m = \frac{d}{1 \text{ mile}} = \text{the } ratio \text{ of } d \text{ to 1 mile.}$$

So that the full expression for the relation $f = \frac{2}{3}m^2$ is :—

$$\frac{\text{height}}{1 \text{ foot}} = \frac{2}{3} \text{ of} \left[\frac{\text{distance}}{1 \text{ mile}}\right]^2.$$

My position, therefore, as regards numerical equations, is this: That the numbers which appear are only short methods of stating pure ratios, and that such short methods are eminently useful in dealing with practical problems, but do not help a student to grasp the fundamental principles of a subject.

There is another simple way in which numerical equations can be deduced from the fundamental ones; viz. by so choosing the standards of measurement that every term may be expressed in terms of the same standard, which may then be omitted, leaving only a relation among the numerical coefficients of that standard.

To enable this to be done, all the standards of subsidiary quantities are so chosen that, when expressed in terms of certain primary standards, their coefficients shall be unity. When this is systematically done, all the standards are usually called *units*, apparently because if you arbitrarily put *unity* for each primary standard, the subsidiary ones will become equal to unity also.

For example, if a foot and a second are chosen units of length and time, a foot per second is the unit of velocity. For, the full expression for a foot per second is $\frac{1 \text{ foot}}{1 \text{ sec.}}$; and if you put 1 foot = 1, and 1 sec. = 1, the fraction $\frac{1 \text{ foot}}{1 \text{ sec.}}$ becomes equal to 1 also.

This plan certainly enables the working numerical equations to be very easily deduced from the fundamental ones, with which indeed they thus become identical in form, but there is great danger lest this fact should make us lose sight of the important fact that they are only special deductions from the higher kind of equation—from the true fundamental equations which exist among the quantities themselves.

DISCOVERY OF ELEPHAS PRIMIGENIUS *ASSOCIATED WITH FLINT IMPLEMENTS AT SOUTHALL.*

A PAPER with the above title was lately read by Mr. J. Allen Brown before the Geologists' Association. It is of more than ordinary interest to geologists since an attempt has lately been made to show that the mammoth became suddenly extinct by the action of a vast flood seemingly universal in its operation, due to some convulsion or cataclysm, which also changed the climate of Northern Europe.

During last year some important drainage works were carried out at Southall, and sections were exposed in the Windmill Lane, a road running from Greenford, through Hanwell, across the Great Western Railway to Woodlake, skirting Osterley Park, as well as in Norwood Lane, leading from Windmill Lane, south-westward.

The remains of the mammoth were discovered in Norwood Lane at the 88-foot contour, about 550 yards from its junction with the Windmill Lane. They were embedded in sandy loam, underlying evenly stratified sandy gravel, with a thin deposit of brick earth, about 1 foot in thickness, surmounting the gravel—in all, about 13 feet above the fossils. The tusks were found curving across the shore or excavation, attached to the skull, parts of which, with the leg-bones, teeth, &c., were exhumed, other bones being seen embedded in one side of the cutting. Probably the entire skeleton might have been removed if the excavation could have been extended, and if there had been appliances at hand for removing the fossils, which were in a soft pulpy condition.

The author obtained some of the bones in a fragmentary state, including parts of the fore-limbs and jaw, with portions of the tusks as well as two of the three teeth found, which were much better preserved. The remains were quite unrolled, and the joints and articulations of the leg-bones and the teeth were unabraded. There can hardly be a doubt, from the report of the workmen, that the bones of the fore-part of the elephant, if not of the entire skeleton, were in juxtaposition.

Several implements were found in Norwood Lane, in close proximity to the remains, and a well-formed spear-head, nearly 5 inches in length, of exactly the same shape as the spear-heads of obsidian until recently in use among the natives of the Admiralty Islands, and other savages, was discovered in actual contact with the bones; smaller spear-head flakes, less symmetrically worked, were also found at this spot. They are formed for easy insertion into the shafts by thinning out the butt ends, similar to those found abundantly by the author at the workshop floor, Acton, and described by him in his recently published work, "Palæolithic Man in North-West Middlesex." Among the implements found at this spot are an unusually fine specimen of the St. Acheul or pointed type, 8 inches long, of rich ochreous colour and unabraded, and a well formed lustrous thick oval implement pointed at one extremity, rounded at the other, about 5 inches in length, also unrolled.

From the adjacent excavations in the Windmill Road several good specimens of Palæolithic work were also obtained, including two dagger implements, with heavy unworked butts, and incurved sides converging to a long point; these were evidently intended to be used in the hand without hafting. Also an instrument characteristic of the older river drift, convex on one side, and slightly concave on the other near the point, and partly worked at the butt. With these were two rude choppers or axes, two points of implements with old surfaces of fracture, a shaft-smoother or spoke-shave, and several flakes.

It is remarkable that most of the principal types of flint implements which characterize the oldest river-drift deposits are represented in this collection from the vicinity of the remains of the elephant.

Mr. J. Allen Brown accounts for the deposit of fossils and associated human relics at this locality by the fact that the underlying Eocene bed rises to within 2 or 3 feet of the surface a few yards west of the spot where the bones and implements were found, while towards the Uxbridge Road and upper part of the Windmill Lane the drift deposits thicken, until at no great distance they have a thickness of 14 to 17 feet. Thus the river drift rapidly thins out, and the upward slope of the London Clay reaches nearly to the surface at about the 90-foot contour. As the level at which the fossils were found (13 feet from the surface) would represent the extent of the erosion and in-filling of the valley which had taken place, it is probable that the higher ground formed by the up-slope of the London Clay then formed the banks of the ancient river; or if another thick bed of drift should be found still further west in a depression of the Tertiary bed such as often occurs, the intervening higher ground would form an island in the stream. In either case a habitable land surface would be formed with shallow tranquil water near the banks, not impinged upon by the current, which afterwards set in the direction of this spot as shown by the coarser slatified gravel above the loamy bed and remains.

The author is thus led to the conclusion either that the carcass of the elephant drifted into the shallow tranquil water near the bank, or else, as seems more probable from the presence of so many weapons near the spot, including the spear-head found with the remains, that the animal was pursued into the shallow water by the Palæolithic hunters and there became bogged. Whatever hypothesis may be accepted, there is no evidence of any greater flood or inundation than would often occur, under the severe climatal conditions which prevailed during the long period that intervened between the formation of the higher brenches of river drift and that of the mid terrace, only 25 to 30 feet above the present river, in which the remains of the mammoth and the extinct Quaternary Mammalia are more frequently met with under similar conditions. Nor does there appear to be any more reason for ascribing the extinction of the great Quaternary Pachyderms to a sudden catastrophe or cataclysm than there is for the extinction of some other Pleistocene animals, such as the great Irish elk, which lived on into, or nearly into, historic times. The difficulty involved in this hypothesis is still further increased by the fact that other animals, such as the reindeer and others of northern habit, as well as southern forms like the hippopotamus, were not