Variable Stars.

Star.	R.A. h. m.	Decl. ° ′			h. m.	
T Cassiopeiæ	0 17·2	55 10 N.	Jan.	19,		*M*
U Cephei	0 52·4	81 16 N.	,,	15,	22 1	*m*
			,,	20,	21 41	*m*
Algol	3 0·9	40 31 N.	,,	18,	2 59	*m*
			,,	20,	23 48	*m*
V Tauri	4 45·6	17 21 N.	,,	15,		*M*
ζ Geminorum	6 57·5	20 44 N.	,,	19,	7 0	*m*
R Canis Majoris	7 14·5	16 12 S.	,,	19,	21 45	*m*
			,,	21,	1 1	*m*
T Canis Minoris	7 27·8	11 59 N.	,,	19,		*M*
W Virginis	13 20·3	2 48 S.	,,	17,	3 0	*M*
δ Libræ	14 55·0	8 4 S.	,,	18,	4 8	*m*
W Scorpii	16 5·2	19 51 S.	,,	17,		*M*
U Ophiuchi	17 10·9	1 20 N.	,,	15,	17 47	*m*
		and at intervals of			20 8	
β Lyræ	18 46·0	33 14 N.	Jan.	16,	6 0	*M*
R Lyræ	18 51·9	43 48 N.	,,	16,		*M*
T Vulpeculæ	20 46·7	27 50 N.	,,	19,	21 0	*M*
Y Cygni	20 47·6	34 14 N.	,,	16,	20 54	*m*
			,,	19,	20 47	*m*
δ Cephei	22 25·1	57 51 N.	,,	17,	6 0	*m*
			,,	18,	21 0	*M*

M signifies maximum; *m* minimum.

DUNÉR ON STARS WITH SPECTRA OF CLASS III.[1]

II.

A SERIES of observations such as ours ought to add at least a little to our knowledge of the development by which the spectra of stars pass from the second class to one of the two sections of the third, especially if these observations are combined with those made of the stars of the two first classes generally, and of our sun in particular; we might even draw conclusions as to the successive development of stars after they have already reached this class. He who sees trees in a forest in different stages of development, some old, some young, some decaying, can at once form an idea of the different stages undergone by each: it is just the same with the observer of the different classes of stellar spectra.

The spectra of the first class are characterized by the almost total absence of all metallic lines excepting those of hydrogen. In spite of that, we cannot doubt for a moment the presence of metallic gases in their atmospheres, for even in the spectrum of Vega we can faintly distinguish the principal rays of sodium, magnesium, and iron. But these gases are probably at such a high temperature that their power of absorption is very slight. But as the star cools and the spectrum approaches the second class, the metallic lines become stronger and more numerous, whilst, strange to say, the lines of hydrogen diminish. Thus the spectrum becomes more and more like that of our sun in its actual state, and at length, as the metallic lines increase, it resembles that of Arcturus.

Up to this stage of development it is unnecessary to consider the two divisions of the third class separately, but after this it becomes indispensable.

In those spectra which at length become III.*a*, the change seems to operate as follows. On account, probably, of the progressive cooling, the metallic lines, especially those of iron, magnesium, calcium, and sodium, become larger, and, besides these, numerous weak narrow lines are seen grouped together, generally in the neighbourhood of the stronger lines. At this stage it is often difficult, if not impossible, to decide, with spectroscopes of small dispersion, whether one sees broad lines or real bands (or flutings). This happens in the spectrum of Aldebaran. The faint lines go on accumulating, until they cannot be separated from one another and occupy broader spaces, and now the spectrum is easily seen to belong to Class III.*a*. At first the bands in the red and orange are the only ones distinctly visible; but later the bands in the green-blue and in the blue become very strong and broad.

While the development of the stars III.*a* was very well known before my researches, former observers have known no star with a spectrum intermediate between II.*a* and III.*b*. Thus, M. Pechüle declares the hypothesis of the co-ordination of the III.*a* and III.*b* classes to be inadmissible. On the other hand, he seems disposed to think that the spectra III.*b* represent a phase, perhaps the last before its total extinction, in the development of each star, and that the passage from type III.*a* to III.*b* takes place suddenly or by a catastrophe, during which the bright lines appear ("Expedition Danoise," pp. 22–25). M. Pechüle seems, however, to consider this hypothesis doubtful, and at length declares that the physical *rôle* of the stars III.*b* is still quite a mystery.

A very simple explanation clears up at least part of this mystery. If the hypothesis which I, in full agreement with M. Vogel, have suggested be correct, the stars intermediate between the second and third classes must necessarily be comparatively rare, considering that this is only a transitory phase of their existence. The general spectroscopic observations of M. Vogel affirm this fact, for amongst the numerous stars examined by him there are only forty-eight whose spectra are denoted by II.*a*!!! II.*a*!! or II.*a*! But as the lines must be very distinctly visible in the spectra of the stars which are on the point of passing from the second class to Class III.*a*, we are obliged to acknowledge that almost all the stars of this category within the zone examined by M. Vogel are among these forty-eight objects. At first sight one might be disposed to seek these stars among those whose spectra are designated by M. Vogel by II.*a* (III.*a*), II.*a*? III.*a*, and III.*a* (II.*a*); but a closer examination shows that although it is not impossible that these spectra may be among these objects, they must be so rare that that is of no essential consequence as regards the question which occupies us.

Amongst these stars there are none which attain the magnitude 4·5, and only fourteen which surpass the magnitude 6·4. All the others are faint objects, and the ambiguous symbols show the difficulty M. Vogel found in recognizing with certainty the details in the spectra, and not that he could not decide with certainty to which of the two contiguous classes a spectrum of which he could easily perceive the details belongs. The correctness of this supposition is, however, proved by the circumstance that certain spectra are designated by III.*a*(III.*b*), or III.*a*? III.*b*. And none will believe that M. Vogel meant to imply that these spectra were in the act of passing from one section of the third class to the other. Besides, one of these stars is R Serpentis, whose spectrum when the star is at the maximum is one of the most strongly marked of III.*a*, according to M. Vogel's earlier researches, and according to mine. But in his general spectroscopic review M. Vogel examined it when its magnitude was only 9·0, and therefore it was easy to doubt, on account of the excessive width of the bands, whether the spectrum might not be III.*b* instead of III.*a*.

Consequently, although I think I am right in admitting that most of these stars belong to the pure type II.*a* or III.*a*, I will nevertheless suppose that a third of them really have spectra intermediate between II.*a* and III.*a*. Their number in M. Vogel's catalogue is 120, and the third is 40, so we should have therefore between the Pole and −25° declination 160 spectra intermediate between II.*a* and III.*a*. I found also by special observations that among the spectra designated by II.*a*!!! II.*a*!! and II.*a*! a fourth part really belong to the intermediate type. Thus there would be in all 200 such spectra, a number evidently much too great. Then, the spectra III.*b* being about fifty times rarer, we should have at most four spectra intermediate between II.*a* and III.*b*, and if only stars of a higher magnitude than 6·0 are reckoned, there would scarcely be one.

But, if we consider the differences between the spectra III.*a* and III.*b*, we shall find that in reality we can scarcely expect to find any spectrum intermediate between II.*a* and III.*b*. As we have seen above, the spectra III.*a* are formed by the exaggeration of the essential characteristics of the spectra II.*a*. There must then be a phase, especially if the star is not very bright, in which one cannot decide to which of the two classes the spectrum belongs. Thus in the spectra III.*b* there are undoubtedly well-marked Fraunhöfer lines—for instance, D, and the narrow band 8, which is probably nothing but the collection of strong lines in the neighbourhood of E, and the very narrow band 5 ($\lambda = 576\mu$) which is almost like a broad line; but all these details are only secondary. The essential characteristics are the three nebulous, very broad flutings, which owe their origin to some carbon compound. If these bands are visible, the spectrum is called III.*b*; if they are not, it is called II.*a*. The only forms intermediate between the spectra of the type of Aldebaran and the normal

[1] Continued from p. 236.

type III.*b* are those in which the bands are more or less faint, or even scarcely perceptible. In fact, I have proved not only that there are spectra in which the principal bands, and especially band 6, are weak on account of the brightness of the stars, but I have found a spectrum which is scarcely a spectrum III.*b* yet, but in which the characteristics of this class are undoubtedly present.

This star is DM. + 38° 3957 = 541 Birm. In its spectrum (Planché, Fig. 6) I have seen a rather broad and well-marked band, whose approximate wave-length is 519μ, and the spectrum terminates abruptly at 475μ. These wave-lengths are, within the limits of probable errors, the same as those of the less refrangible ends of bands 9 and 10 in the spectra III.*b*. Once I thought I perceived a very faint trace of light beyond 475μ, and in the best atmospheric conditions I caught a glimpse of faint traces of the bands 4 and 6. Unfortunately the star is only of the eighth magnitude, so that only few details of its spectrum can be seen with a telescope like ours. Nevertheless, what I did see seems to me of some importance in explaining the development of a spectrum II.*a* into III.*b*.

If this spectrum be compared with those of other stars of the same or even of a lower magnitude, such as 145 Schj. DM. + 34° 56, DM. + 36° 3168, it is at once seen that in the former the principal bands are still in a very low stage of development, and if the bands had only been a little paler nothing unusual would have been seen in the spectrum under ordinary atmospheric conditions. The aspect of this star seems to prove what I said above, that there is, properly speaking, no intermediate state between the spectra II.*a* and III.*b*, but that the passage from one to the other is already accomplished before the first traces have been perceived.

But there is still one more circumstance deserving of attention, which may perhaps lead to the knowledge of other spectra which are still nearer to the critical point; that is, the very strong absorption of the more refrangible rays, which makes the whole spectrum very short, and gives to the star itself its bright orange colour. We know that there are many stars of a deep colour and with short spectra, but otherwise not striking; they ought to be examined from time to time with very powerful microscopes, for amongst these will be found, I believe, the new spectra III.*b*.

There are other spectra, which, although they undoubtedly belong to Class III.*b*, have not, it appears, reached their full development. The least faint of these stars is that known as 7 Schj. Before my researches, nothing had been published regarding this spectrum except this short remark of D'Arrest, "Irregular spectrum, probably type IV." (*Vierteljahrschrift der Astr. Ges.* ix. Jahrg. p. 255). This spectrum presents the characteristics of III.*b* very pronounced; only band 5 is invisible, and band 6 is so faint that at first sight the spectrum has not the aspect characteristic of well-developed spectra of this class. It is for this reason that D'Arrest would say nothing positive regarding this star. If the spectrum of 541 Birm. represents the first step in the passage of a star to Class III.*b*, this star doubtless represents the second step. Band 6 is the least developed of the three principal ones. Although the spectrum of this star is pretty bright, band 5 is not visible, whilst band 4 is well visible, and is also perceived in the spectrum of 541 Birm.

In the spectrum of 19 Piscium (Fig. 4 on the map), which is one of the most magnificent in other respects, band 6 is still considerably fainter than the other two principal bands, whilst in that of 152 Schj. (Fig. 3) it is quite as pronounced as band 10, and almost as pronounced as band 9. This last spectrum is in an advanced stage of development; but in spite of that, band 4 is not stronger than in the spectrum of 7 Schj., and rather fainter than that of 19 Piscium. The same relation is repeated in other spectra of this class, so that sometimes band 4 is very visible in an otherwise less developed spectrum, but invisible in more strongly marked spectra, and in the spectra of brighter stars of this class there are in the same way very faint bands, 7 and 8. But band 4 is in itself very pale; it is the deep sodium line which makes it remarkable, and the bands 7 and 8 are probably only groups of Fraunhöfer lines.

It is therefore very probable that the more or less easy visibility of these bands is no indication as to the phase of development in which the star is. There is, on the contrary, reason to believe that the strengthening of these lines, and also of the other principal lines of the spectrum (except those of hydrogen, which grow fainter during the passage of a star to Class III.) is a process of relatively small importance which goes on whilst the star still undoubtedly belongs to Class II.*a*; and even when this is accomplished there is still nothing to show whether the star will become III.*a* or III.*b*, unless perhaps in those which tend towards Class III.*a*, the line, or rather group of lines, with wave-length 616, is very well marked, which seems not to take place in the spectra III.*b*. But in the stars which tend towards the latter class the violet rays are already very much absorbed, and the stars are therefore of a deep orange.

If we pass on to consider the ulterior development of the star, it is evident that as it cools further it at length reaches a temperature at which the carbon which must be present in abundance, either in its atmosphere or under some form in its photosphere, can combine with hydrogen or some other element to give the so-called hydrocarbon spectrum. After that, the spectrum appears cut by a broad faint band with the wave-length 516μ, and by another still paler at 473μ, and the parts of the spectrum beyond this are very faint. But gradually these two bands increase in intensity, and at the same time the band 563μ is perceived, at first very faintly, and gradually becoming stronger. At this stage the narrow band 576μ is developed, and finally the three principal bands are nearly of equal intensity, and the spectrum shows all the characteristic details. It would be useless to attempt to discuss the moment at which the secondary bands in the red and orange make their appearance, as no facts on the subject are known.

It is doubtless very remarkable that in the spectra III.*b* no trace of the carbon band with the wave-length 618·7μ is seen, which is so brilliant in Plücker's tubes containing hydrocarbon. This is, however, in perfect analogy with what is seen in the spectra of comets, which owe their appearance to the same carbon compound as the stellar spectra III.*b*, and there are analogies also for the other bands. Thus the band 563μ is often very weak even in the bright comets, and the band in the green is always the strongest both in comets and stars. The band in the blue is sometimes pretty faint in cometary spectra, whilst in the stars it is only a little fainter than the band in the green; but we must remember that it is situated in a very faint part in the spectra of the stars. It is therefore very possible that a little dimness should render the remaining light entirely imperceptible. In this perhaps there is no diversity between comets and these stars. The violet bands are very faint in Plücker's tubes, but strong in the flame of alcohol. A trace of them has been seen in the spectra of the brightest comets. In very brilliant, not too red stars III.*b*, there is also a violet zone, terminating at the wave-length 430μ, of which there is a band at the position of the first and the second of these bands in the spectra of these stars.

We will now pass on to consider the changes which take place in stars of Class III. after their spectra have completely developed. As the cooling goes on, they necessarily grow dimmer and dimmer, and at length become extinct. Either the bands in their spectra must increase in width until at last the shining intervals disappear, or else, the bands keeping their same width, the whole spectrum grows fainter. Certainly we see that there are stars whose bands are enormously broad, but none the breadth of whose bands surpasses that of the bright zones.

I think, therefore, we can hardly accept the first hypothesis, but there are reasons which give very valuable support to the second. We know that the weakness of the light in the solar spots is, in the first place, caused by a general obscuration of the spectrum, and that the enlargement of the Fraunhöfer lines has very little to do with it. Besides, I have examined, on different occasions, between the maximum and the minimum, the spectra of several variable stars of Class III., and found that there was no widening of the bands sufficient to explain the weakening of the stars. There is no doubt a remarkable analogy between the spectra of the sunspots and those of the stars of Class III., and one which we have no cause to be surprised at. For, on account of the relatively low temperature of these stars, it is very probable that their surfaces are in great part covered with formations similar to our sunspots, and the absorption-bands found in their spectra are no argument against this analogy. They prove only that chemical compounds may be formed and maintained in the atmospheres of these stars, which is not possible in our sun, not even in the masses of relatively low temperature of which the spots consist.

Before laying down my pen I must remark that the induction by which I arrived at these conclusions does not prove that the spectrum of each star commences with Class I. and finishes with

Class III. The development might just as well be in inverse order, though we have important reasons for believing it is not so.

The astronomy of the future must decide between these two alternatives. My object in undertaking this work was to facilitate this decision by giving as exact descriptions as possible of the spectra presented by the different stars of Class III. in the year 1880.

THE ART OF COMPUTATION FOR THE PURPOSES OF SCIENCE.[1]

II.

SOME few problems in astronomy and certain theories in pure mathematics require more than seven figures to be calculated. In these cases a large arithmometer is generally the most convenient. Ten-figure tables of logarithms may be obtained second-hand; or the required logarithms must be calculated.

The tables of Vlacq, re-edited by Vega in 1749, 1794, and 1797 are somewhat difficult to obtain and cumbrous to use. The logarithms of numbers up to 101,000 are given to ten figures with first and second differences. Thus to find log 10 542 482 375, from the table directly

$$\begin{array}{lrlr}
\log 10\,542 = & \cdot0229\,230\,119 & \Delta_1 = & 411\,946 \\
 & 198\,712{\cdot}3 & & 482\,375 \\
 & 5 & & \overline{1\,647\,784} \\
\text{log required} & \overline{\cdot0229\,428\,836\,3} & & 329\,557\ + \\
 & & & 8\,239\ + \\
 & & & 1\,236\ + \\
 & & & 288\ + \\
 & & & 20\ - \\
 & & & \overline{198\,712\,3\,1}\ \text{subtracted.}
\end{array}$$

The true log of 10 542 482 375 is ·022 942 883 626 562.

$$\Delta_2 = 40$$

$$\frac{\cdot48(\cdot48 - 1)(-40)}{2} = 4{\cdot}992.$$

In default of Vega, or if more places are required, the logarithm must be calculated, and this is by no means such a serious affair as one is led to think by the ordinary books on algebra. I am much indebted in what follows to the article by Mr. J. W. L. Glaisher on logarithms in the new edition of the "Encyclopædia Britannica," to which I refer my readers for further particulars in theory, restricting myself to practical details.

The easiest way to calculate a table of logarithms absolutely *de novo* would be by the method of differences, with some mechanical assistance, such as the difference-engine of Babbage or of Scheutz. It seems unlikely that larger tables will be calculated than those already in existence, since the cost increases with great rapidity. Mr. Sang has, however, recently calculated independently the logarithms of numbers from 100000 to 200000, where the ordinary tables are weakest.

Briggs used at least two methods for the calculation of logarithms which depended upon the extraction of a succession of roots. For instance, by taking the square root of 10 fifty-four times he found $\log 1{\cdot}(0)^{15}1\ 278\ 191\ 493$ to be $\cdot(0)^{15}0\ 555\ 111\ 512$. Whence assuming that very small numbers vary as their logarithms, $\log 1{\cdot}(0)^{15}1 = 555\ 111\ 512/1\ 278\ 191\ 493$, or $\log 1{\cdot}(0)^{15}1 = 0{\cdot}43\ 429\ 448 = M$, the modulus. And if x be small, $\log 1{\cdot}(0)^{15}x = x \times 0{\cdot}43\ 429\ 448$. To find log 2 he extracted the square root of the tenth power, 1024/1000 forty-seven times, and found $1{\cdot}(0)^{15}1\ 685\ 160\ 570$, which multiplied by M gave $(0)^{15}0\ 731\ 855\ 936$. This multiplied by 2^{47} gave log 1·024; adding 3 and dividing by 10 gives log 2. Another more simple method was to find a series of geometrical means between two numbers, such as 10 and 1, the logarithms of which are known. After taking 22 of these roots, log 5 is found to be 0·69897.

It was soon found that logarithms could be more easily calculated by the summation of various series, and many great mathematicians, such as Newton, Gregory, Halley, Cotes, exercised their ingenuity in discovering those most suitable for the purpose.

Though for practical purposes the use of series has been almost superseded, three very simple ones are still occasionally useful :—

$$\log(1 \pm x) = M\left(\pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \frac{x^5}{5} - \right)$$

which converges rapidly if x be small. M is a number depending upon the system of logarithms adopted, and constant for each system. If M be 1, the system is called the Naperian, or natural one; and if M = 0·434 &c., the system is the common one. Unless otherwise stated M will be assumed to be 1, or the logarithms will be natural ones.

Thus to calculate $\log 1{\cdot}1 = 1 + \frac{1}{10}$, omitting M :—

$$\log 1{\cdot}1 = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} - \frac{1}{40000} + \frac{1}{500000} - , \&c.,$$
$$= 0{\cdot}1003\ 3534 - 0{\cdot}0050\ 2517 = 0{\cdot}0953\ 1017.$$

Suppose x be small, $\log(1 \pm x) = \pm Mx$ nearly. Thus if $\log 1{\cdot}(0)^{10}9$ be required to twenty decimals, it is

$$(0)^{10}9 - \frac{1}{2}(\cdot9 \times 10^{-10})^2,$$

or the error caused by omitting this and all subsequent terms is only 4 in the twenty-first decimal place. Using common logarithms the multiplication by M reduces the error by one-half. This result is of great importance in calculating logarithms by Flower's method, since the factors which have to be dealt with are only half the number of decimal places in the required logarithm.

Writing $\frac{1}{x}$ for x in the above series, we obtain—

$$\log(1 + x) - \log x = M\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \frac{1}{5x^5} - \right)$$

which converges rapidly when x is large. Various artifices may be used to render x large, even when the number the logarithm of which is required is small. Thus, Prof. J. C. Adams has calculated (NATURE, vol. xxxv. p. 381) log 2, log 3, log 5, log 7, $\frac{1}{M}$ and M to 270 places of decimals.

Another very valuable series is—

$$\log(a \pm x) = \log a \pm 2M\left\{\frac{x}{2a + x} + \frac{1}{3}\left(\frac{x}{2a + x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a + x}\right)^5 + \&c.\right\}$$

Thus, supposing log 219 known, to calculate log 2198 :—

$$\log 2198 = 7{\cdot}6916\ 5682\ 2810 + 2\left\{\frac{4}{2194} + \frac{1}{3}\left(\frac{4}{2194}\right)^3 + \&c.\right\}$$

$$\begin{array}{lrl}
 & \cdot0036\ 4630\ 8113 & \qquad \frac{4}{2194} = \cdot0018\ 2315\ 40565 \\
 & 4039 & \\
\log 2198 = & \overline{7{\cdot}6593\ 0313\ 4962} & \qquad \frac{2}{3}\left(\frac{1823}{10^6}\right)^3 = \cdot0^8 4039
\end{array}$$

Using common logarithms, the third term of the series is $< \frac{1}{27{\cdot}6}\left(\frac{x}{a}\right)^3$, that is less than 5 in the ninth place when $\frac{x}{a} < \frac{1}{200}$. Hence, with a table giving the logarithms of 100–1000 to eight figures the third term may be neglected, or the required difference is $\pm \frac{2Mx}{2a + x}$, or, writing $\log(a + x) - \log a = y$, $x = \frac{2ay}{2M - y}$.

The given numbers may also be broken up into factors by the aid of such a table as Burkhard's, which gives the factors of all numbers up to 3,036,000. The logarithms of the factors may then be found from tables and added together. Of all tables for this purpose, that of Wolfram is the most valuable; it gives the natural logarithms to forty-eight places of *all* numbers up to 2200, and of all which are not easily divisible up to 10,009.

The multiplication by M to convert into common logarithms is tedious, and it is frequently better to dispense with it in heavy calculations. If necessary, a table of the first ninety-nine multiples of M should be prepared, and Oughtred's short method of multiplication used.

If any of my readers desire to test themselves and their tables

[1] Continued from p. 239.