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THE MATHEMATICAL THEORY OF PERFECTLY ELASTIC SOLIDS.

An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids; with a Short Account of Viscous Fluids. By William John Ibbetson, M.A. (London: Macmillan and Co., 1887.)

IT is strange that students should have had to wait till the present time for a systematic English text-book on the mathematical theory of elastic bodies. The want has been decidedly felt at Cambridge since the introduction of the subject into the schedule for the Mathematical Tripos in 1873; and though parts of Thomson and Tait's treatise on natural philosophy, and the reprint of Green's papers, had already brought a large amount of useful matter into an accessible form for those who had not time or opportunity to read the original memoirs, still it was found that learners, naturally looking for some compendious account of the whole subject, generally fell back upon M. Lamé's treatise.

The book at present under notice will supply this want satisfactorily. The plan on which it has been written is excellent in idea, and has on the whole been followed out well, though perhaps there is here and there some want of proportion, as for instance in the elaborate and purely mathematical details of Chapter V.

It is, no doubt, a difficult matter to decide what results of mathematical analysis should be introduced without proof in a treatise on mathematical physics, and there is little question that, as a matter of convenience to the reader, it is wiser to err on the side of assuming too little knowledge rather than too much. On the other hand, wherever questions of pure mathematics are introduced and discussed at length, they should be such as have a direct bearing on important parts of the physical subject. Now the general forms of the dynamical equations for an elastic body in terms of curvilinear co-ordinates, which are established in Chapter V. after a considerable amount of preliminary analysis, are so complicated as to be practically valueless. Indeed, the one case referred to is dismissed in a single paragraph. The special forms for polar and semi-polar co-ordinates, often to be used with advantage, may be very much more simply established independently.

To return to the general plan of the book. It commences with a short preliminary chapter, headed "Properties of Elastic Solids," in which, after showing that the subject cannot be profitably considered from the point of view of molecular structure, the author defines the ideal solid which must for purposes of analysis replace the real body.

In Chapter II. the general properties of strain are treated very clearly and at considerable length. A little more consideration might perhaps have been given with advantage to finite homogeneous strain. No fewer than three quadrics are introduced for the purpose of putting results into a geometrical form, viz. the strain ellipsoid, the elongation quadric, and the position ellipsoid; while in the succeeding chapter, on "The Analysis of Stress," four surfaces—the first, second, third, and fourth stress quadrics—are used for a similar purpose. It can hardly be doubted that so great a number of surfaces will tend rather to confusion than to that clearness of conception of the properties of strain and stress for which they are presumably introduced.

The nature and mode of specification of stress is carefully expounded in Chapter III., and the dynamical equations to be satisfied throughout the body and over the boundary are obtained in terms of the stress-components. Attention should be called to a statement in § 153 of this chapter, as likely to mislead the student. It is to the effect that "the component stresses are to be considered as small quantities of the first order." Though in a certain sense this is true, it is not true that the ratio of the stress per unit area to, say, the weight per unit volume of the body is a small proper fraction, and this surely is the strict sense the words should bear.

The next chapter, on "The Potential Energy of Strain," is excellent. The method is similar to that used by Thomson and Tait; and the successive simplifications introduced into the expression for the potential energy of the strained body by considering successively greater degrees of symmetry of structure, leading up to perfect isotropy, are well shown. It is also pointed out that from the definition of an isotropic body, its potential energy is necessarily a function of the invariants of the strain, thus reducing the number of independent elastic constants in this case at once to two. Having thus arrived at a definite conception of the isotropic elastic solid, the author expressly limits all the investigations that follow to the case of such a body. From the expression obtained for the potential energy, forms are deduced for the stress components in terms of the strain components and the elastic constants, and thence finally the dynamical equations are obtained in terms of the displacements.

In Chapter V., as already stated, these equations are thrown into new forms, and the remainder of the book is devoted to their solution under various conditions as to the nature of the applied forces and tractions and the form of the body.

As an introduction to the consideration of particular questions the following five general theorems are proved, viz.:—

"(i.) that a state of strain cannot be maintained unchanged without the action of applied forces or surface tractions:

"(ii.) that the state of strain maintained by a given system of equilibrating applied forces and surface tractions is therefore perfectly determinate:

"(iii.) that the most general free motion of an elastic solid consists of a number of superposed harmonic oscillations of the particles about their natural positions:

"(iv.) that the most general motion of such a body under the action of an equilibrating system consists of a number of superposed harmonic oscillations of the particles about the equilibrium positions that would be maintained by the system:

"(v.) that a system of applied forces varying as a simple harmonic function of the time gives rise to forced harmonic oscillations of the particles of the same period about their natural positions."

The proofs given of these theorems, and especially of the third, are rather unnecessarily long; but, with a view to avoiding repetition later on, it is certainly convenient

to have them established as a foundation from which to start.

The problem of free vibrations is first treated, and as an example the propagation of plane waves of (i.) normal, (ii.) tangential, displacement is investigated. It is a pity that the author has not here taken the opportunity of illustrating some previous remarks on the discontinuity of the forms of the strain and stress components which necessarily accompanies a change in the nature of the medium, by considering the question of the reflection and refraction of plane waves.

The general form of the solution of the equations for forced vibrations is next investigated. Then follows the general question of equilibrium. As a simple example the case of a cylindrical tube under external and internal normal pressures is first treated. It is almost annoying to find the solution of this time-honoured question obtained by starting from the general equations, and whittling them down till the very simple conditions are fulfilled. The equilibrium of a solid sphere, with either surface tractions or displacements given, is treated exactly as in Thomson and Tait's work. The chapter of general solutions closes with an account of Airy's general method for plain stress, with a couple of examples. The printing was unfortunately so far advanced that this had to be left; though before the book appeared the author had himself shown, in a communication to the London Mathematical Society, that these examples of Airy's are faulty, and that the method applies only to a very limited class of cases.

Chapter VII. consists mainly of a capital exposition of the solution of St. Venant's problems of the torsion and flexure of prisms. These problems are probably, from a practical point of view, the most important for which an exact solution has been obtained. The author brings out well the bearings of the nature of the solutions on practical questions of construction. The elastic equilibrium and small motions of wires, whether straight or curved, are deduced directly from the results of St. Venant's problems. In connection with this part of the subject, certain interesting questions of stability, due to Mr. Greenhill, are discussed.

Some cases of the equilibrium and vibrations of plates and shells are considered in Chapter VIII. For the equilibrium of a plate of uniform thickness under a system of surface tractions parallel to its faces and acting on its edges, a solution is obtained by analysis very similar to that used in St. Venant's problem. The case of a thin plate under the action of applied faces satisfying certain conditions is quoted from Thomson and Tait again.

Two short chapters headed "Impact" and "Viscosity" complete the volume. The former consists of the solution of two problems, one of which, as the author implies, has nothing to do with impulsive change of motion. Indeed, as is well known, the exact treatment of the impact of elastic bodies involves difficulties, even in comparatively simple cases, which have not yet been overcome. In the last chapter the alteration in the form of the dynamical equations is determined, which results from supposing the shearing stress to vary partly as the shear and partly as its rate of change.

Having thus given some account of the plan of the book and the way in which it has been carried out generally, we may offer some remarks on matters of detail. It may be said at once that as regards accuracy there is a good deal to be desired. The table of errata might have been tripled, and would not then have contained all the misprints. In §§ 299, 306, the wholesale omission of signs of summation in the equations makes the analysis, as given, incorrect; and there is little doubt but that any-one to whom the matter treated was new would be completely baffled. The inaccuracies, moreover, are not confined to mere misprints. There are one or two positive mistakes in the mathematics. Thus at the bottom of p. 58 it is implied that some condition is necessary in order that a family of surfaces, $f(x, y, z) = \xi$ (an arbitrary parameter), may have a system of continuous curves cutting them at right angles; and in a note at the foot of p. 298 it is stated that, supposing this (entirely imaginary) condition satisfied, two other systems of surfaces can always be found cutting each other and the former surfaces everywhere at right angles. Now the three parameters of such a triple system of surfaces have to satisfy three independent partial differential equations, and hence no one of the three can be taken arbitrarily. Statements and reasoning are, in several passages, founded on this erroneous conception. Closely allied with this is the construction given in § 216 for tubes of stress. It is here practically assumed that a given continuous system of curves can always be cut at right angles by a family of continuous surfaces.

An appendix at the end of Chapter II., on "The Geometry of Strains," might have been omitted with advantage. It has no very obvious connection with the preceding chapter, but is devoted to an apparently new classification of vector quantities, in which a velocity and a force are the types of the one group, while an angular and a couple are those of the other! Again, in §§ 270, 271, the solution of a physical problem is made to appear to depend on the choice of an origin. The question treated is the free normal vibrations of a plate; and, after using d and $-d'$ to denote the abscissæ of the two faces, and making the result appear to depend on d/d', the question is *simplified* by taking the origin midway between the faces. Indeed, frequently throughout the book one is reminded of Clerk Maxwell's remark on "the state of a mind conscious of knowing the absolute position of a point."

These slips, such as they are, and an occasional obscurity of language, are but slight blemishes on a valuable book. A friendly but independent criticism of the proof-sheets while the book was passing through the press might have removed them all, and no doubt will in a new edition.

The figures throughout are excellent.

THE VOLCANIC AND CORAL ISLANDS OF THE SOLOMON GROUP.

The Solomon Islands: their Geology, General Features, and Suitability for Colonization. By H. B. Guppy, M.B., F.G.S., late Surgeon R.N. (London: Swan Sonnenschein, Lowrey, and Co., 1887.)

SURGEONS in Her Majesty's navy are favoured beyond most men in the possession of abundant leisure and freedom from many of the common cares of