yellow, orange, red. A little later, the same night, on the brow of the ridge, in the faint mist which rose in masses, the bow was again seen vividly, against a background of trees; the bow being within 40 paces of the observer.

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September 23.

Destruction of Young Fish.

MAY I call your attention to the wholesale destruction of young fish which is carried on to a great extent round our coast? A few facts may not be out of place. Recently I have been visiting a small fishing village on the east coast, and have carefully noted the amount of young fish rejected by the fishermen on their return from trawling and shrimping. For example, from 4½ pecks of shrimps no less than 793 flat-fish (dabs, soles, and turbot) were thrown on the beach useless; to this must be added about 200 whiting and an amount of young cod, herring, and skate beyond my power to count. Surely something can be done to remedy this! It is well known to the fishermen that the net does not injure the fish; so that before landing, if the net was roughly examined, all young fish could be thrown into the water again.

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ON HAMILTON'S NUMBERS.

FOLLOWING in the footsteps of Hamilton in his Report to the British Association, contained in the Proceedings for the year 1836, we may arrive at a solution, in a certain sense the simplest, of a problem in algebra the origin of which reaches back to Tschirnhausen, born 1651, deceased 1708. Every tyro knows how a quadratic equation, and all equations of a superior degree thereto, may be transformed into another in which the second term is wanting. Tschirnhausen showed that a cubic equation, and all equations superior in degree to the cubic, might be deprived of their second and third terms by solving linear and quadratic equations. Then over a century later Bring, of the University of Lund, in 1786 showed that every equation of the 5th, or any higher degree, might be deprived of its first three terms by means of solving certain cubic, quadratic, and linear equations.¹ What, then, it may be asked, is the law of the progression of which the three first terms are 2, 3, 5? What is the lowest degree an equation can have in order that it may admit of being deprived of four consecutive terms by aid of equations of the 1st, 2nd, 3rd, and 4th degrees, or more generally of i consecutive terms by aid of equations of the 1st, 2nd, 3rd, . . . and i th degrees, *i.e.* by equations none of a higher degree than the i th?²

¹ In a letter to Leibnitz (1677), which I have not seen, and the *Acta Eruditorum* for 1683.

² How the elevation of the degree of the equation to be transformed makes it possible to abolish a greater number (μ) of terms by an auxiliary system of equations of degrees none exceeding μ' will be understood if we consider the cases of a quintic and quartic.

Supposing $(x, 1)^5$ to be a given quintic, on writing

$$\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon = 0$$

we obtain, by elimination of x , $(\alpha, \beta, \gamma, \delta, \epsilon)^5 = 0$, and any solution of his equation will enable us, by a well-known process, to find x by a linear equation.

If we select any letter, α , of the five we may equate it to a linear function of $\beta, \gamma, \delta, \epsilon$, so as to obtain

$$\gamma^5 + (\beta, \gamma, \delta, \epsilon)^2 \gamma^3 + (\beta, \gamma, \delta, \epsilon)^3 \gamma^2 + (\beta, \gamma, \delta, \epsilon)^4 \gamma + (\beta, \gamma, \delta, \epsilon)^5 = 0.$$

If in this equation we can find any system of ratios $\beta : \gamma : \delta : \epsilon$ such that $(\beta, \gamma, \delta, \epsilon)^2 = 0$, and $(\beta, \gamma, \delta, \epsilon)^3 = 0$, we can find γ by solving a trinomial quintic, and therefore a system of admissible ratios $\alpha : \beta : \gamma : \delta : \epsilon$ becomes known.

All that is requisite therefore is to be able to obtain any point whatever of intersection of two given quadratic and cubic surfaces represented by $(\beta, \gamma, \delta, \epsilon)^2$ and $(\beta, \gamma, \delta, \epsilon)^3$ which obviously may be done by first finding a point (any point) in the quadratic surface (which only necessitates solving some quadratic equation or other); second, at this point drawing a right line (either one of a pair) lying on the surface, which may be effected by a well-known method involving only the solution of a quadratic; and third, finding any one of the three intersections of such line with the cubic surface.

Thus, then, by solving quadratic and cubic equations a quintic may be

In the 100th volume of *Crelle's Journal* (1886) I have shown that the progression continued as far as the case of eight terms being abolished is as follows—

2, 3, 5, 10, 44, 905, 409181, 83762797734.

These, with the exception of the three first, are not exactly what I call Hamilton's numbers, but serve to lead up to them.

Hamilton's numbers are—

2, 3, 5, 11, 47, 923, 409619, 83763206255, . . .

I will endeavour to explain wherein the difference consists between the two series.

Whilst it is true that four terms may be abolished in an equation of the 10th degree without solving equations beyond the 4th degree, there is this difference in favour of equations of the 11th or any higher degree, viz. that fewer biquadratics will be required for them than in the case of an equation limited to the 10th degree. And so in general whether we take, as our inferior limit to the degree of the equation to be transformed, the i th number in the upper series or the i th number in the lower one—whilst in neither case it will be necessary to solve any equations of a degree exceeding i —the total system in the latter case will be of a simpler character than in the former.

The numbers which I have named in honour of Hamilton may be obtained by a process exhibited in the table below—

I	O	O	O	O	O	O	O	O	O
1	1	1	1	1	1	1	1	1	1
	2	3	4	5	6	7	8	9	10
	1	5	9	14	20	27	35	44	54
		6	15	29	49	76	111	154	205
			5	21	50	99	175	286	441
			4	26	76	175	350	631	1051
			3	30	106	281	631	1351	2551
			2	33	139	420	1051	2251	4251
			1	35	174	594	1645	3751	7251
				36	210	804	2449	5751	11251
				&c.	&c.	&c.	&c.	&c.	&c.

We may now isolate the greatest figure which occurs in each column, and in this may we obtain the numbers 1, 1, 2, 6, 36 . . . which I call hypotenusal numbers; then adding these numbers together and increasing each sum so obtained by unity we arrive at the so-called Hamilton's numbers, viz. 2, 3, 5, 11, 47, . . . Now the question arises as to how they may be calculated; for obviously the crude method above given will be impossible to carry out in practice beyond the first few numbers in the scale. The method of generating functions—of which the idea occurred first to my coadjutor Mr. James Hammond, which certainly ought not to, and probably in the long run could not, have escaped me—leads to a wonderfully beautiful law, by means of which these numbers may be derived successively each from those that go before, just as is the case with Bernoulli's numbers.

The simplest and best mode of proceeding is as

deprived of three consecutive terms. But not so a quartic; for in the case of a quartic we could not (with any real advantage) use a subsidiary equation of a higher degree than the 3rd, we should thus have only three letters, β, γ, δ , instead of four in the equation in γ , and to make $(\beta, \gamma, \delta)^2 = 0$, $(\beta, \gamma, \delta)^3 = 0$, simultaneously, is the problem of finding an intersection of a quadratic and a cubic curve, which necessitates the solution of an equation of the 6th degree.

In the case of the quintic it may be well to notice that the ratios $\alpha : \beta : \gamma : \delta : \epsilon$ will not be all real, and consequently the trinomial quintic into which the original one has been transformed will not have its coefficients real, unless the quadric surface is a hyperboloid of one sheet (since it is only that species of quadric surfaces which contains real straight lines); and I have shown in my paper in *Crelle* that this is the case then, and then only, when the original quintic has four imaginary roots.

² If the number of equations of degrees $i, i-1, i-2, \dots$ to be solved in the one case reckoned in descending order are $a, b, c, \dots, l, \dots$ and in the other $a', b', c', \dots, l', \dots$ respectively, if l, l' are the two first corresponding numbers which are not identical l' will be less than l .

follows:—Look at the successive lines of figures in the table and write

$$\begin{aligned} 1 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 + \dots &= F_0 \\ x + x^2 + x^3 + x^4 + x^5 + \dots &= {}^1F_0 = F_1 \\ 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots &= {}^1F_1 = F_2 \\ x^2 + 5x^3 + 9x^4 + 14x^5 + \dots &= {}^1F_2 = F_3 \\ 6x^3 + 15x^4 + 29x^5 + \dots &= {}^2F_2 = F_3 \\ 5x^3 + 21x^4 + 50x^5 + \dots &= {}^1F_3 \\ 4x^3 + 26x^4 + 76x^5 + \dots &= {}^2F_3 \\ 3x^3 + 30x^4 + 106x^5 + \dots &= {}^3F_3 \\ 2x^3 + 33x^4 + 139x^5 + \dots &= {}^4F_3 \\ x^3 + 35x^4 + 174x^5 + \dots &= {}^5F_3 \\ 36x^4 + 210x^5 + \dots &= {}^6F_3 = F_4 \end{aligned}$$

and so on.

Then evidently

$${}^1F_i = (1-x)^{-1} F_i - x^i$$

and in general

$${}^jF_i = (1-x)^{-j} F_i - x^i$$

and consequently calling the i th number in the hypotenusal series 1, 1, 2, 6, 36, a_i , and bearing in mind that ${}^aF_i = F_{i+1}$, we shall have

$$\begin{aligned} F_{i+1} - (1-x)^{-a_i} F_i \\ = -x^i \{ 1 + (1-x)^{-1} + (1-x)^{-2} + \dots + (1-x)^{-a_i+1} \} \\ = x^i \{ (1-x) - (1-x)^{-a_i+1} \} \end{aligned}$$

which obviously regarded as an equation in differences of the 1st order in F_i gives the means of expressing F_{i+1} as a function of x and $a_i, a_{i-1}, \dots, a_0$, and consequently must enable us to express all the coefficients in F_{i+1} , of which the first is the hypotenusal number a_{i+1} , in terms of all the hypotenusal numbers of lower order. But what is surprising and unexpected is that, as we shall in a moment see, the relation obtained is expressed by an immediate equation between the sums of the hypotenusals 1, 1, 2, 6, 36, ..., each increased by 2, i.e. by an equation between the *ipsissimi* numbers of Hamilton augmented by unity.

In fact, multiplying each side of the equation by $(1-x)^{S_i+1}$, where

$$S_{i+1} = a_0 + a_1 + a_2 + \dots + a_i$$

(so that $S_1 = 1, S_2 = 2, S_3 = 4, S_4 = 10, S_5 = 46, \dots$), it becomes

$$\begin{aligned} (1-x)^{S_i+1} F_{i+1} - (1-x)^{S_i} F_i \\ = x^{i-1} (1-x) \{ (1-x)^{S_i+1} - (1-x)^{S_i} \} \end{aligned}$$

which equation, it may be noticed, proved for all values of i down to 1 may be extended also to $i=0$, provided we make $S_0=0$.

Accordingly, giving i all values down to 0 inclusive, we shall easily obtain by addition

$$\begin{aligned} (1-x)^{S_i} F_i = 1 + x^{i-2} (1-x)^{S_i+1} - x^{-1} (1-x) \\ + x^{i-3} (1-x)^{S_{i-1}+2} + x^{i-4} (1-x)^{S_{i-2}+2} + \dots \\ + x^{-1} (1-x)^{S_1+2}, \end{aligned}$$

or which is the same thing

$$\begin{aligned} (1-x)^{S_i} F_i - 2 + x^{-1} - x^{i-1} (1-x)^{S_i+1} \\ = x^{i-2} (1-x)^{S_i+2} + x^{i-3} (1-x)^{S_{i-1}+2} + \dots \\ + x^{-1} (1-x)^{S_1+2}. \end{aligned}$$

Hence, equating the coefficients of x^i and using $\beta^k q$ in general to signify $\frac{q(q-1)\dots(q-k+1)}{1.2\dots k}$, if we call

$1 + S_i = H_i$ we obtain

$$\begin{aligned} a_i + S_i + 1 = H_{i+1} \\ = \beta^3(H_i + 1) - \beta^3(H_{i-1} + 1) + \beta^4(H_{i-2} + 1) - \dots \\ + (-)^{i+1} \beta^{i+1}(H_1 + 1) \end{aligned}$$

And on calling $1 + H_i = E_i$ this equation becomes

$$\begin{aligned} 1 - \beta E_{i+1} + \beta^2 E_i - \beta^3 E_{i-1} + \beta^4 E_{i-2} - \dots \\ + (-)^{i+1} \beta^{i+1} E_1 = 0. \end{aligned}$$

This relation between the sharpened Hamiltonian numbers (i.e. these numbers increased by a unit) is in a slightly different form from the result obtained by Hammond. By aid of this formula the values of the successive numbers can be calculated with wonderful facility. The series of them commencing with 1, which although not properly speaking a Hamiltonian number, belongs to the class $S_i + 1$, have been found to be

$$1, 2, 3, 5, 11, 47, 923, 409619, 83763206255, \\ 3508125906290858798171, \\ 6153473687096578758448522809275077520433167, \dots$$

Thus *ex gr.* having found $H_6 = 923$ and all the Hamiltonian numbers inferior to it, we have

$$\begin{aligned} H_7 = \frac{924 \cdot 923}{1 \cdot 2} - \frac{48 \cdot 47 \cdot 46}{1 \cdot 2 \cdot 3} + \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ = 426426 - 17295 + 495 - 6 \\ = 409619. \end{aligned}$$

I have alluded to Bernoulli's numbers as a parallel case to that of Hamilton's in so far as they too are subject to a scale of relation by which each can be expressed in terms of those of a lower order than itself.

If we use

$$B_1, B_2, B_3, \dots$$

to signify as usual the Bernoullian numbers

$$\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \dots$$

and write

$$\begin{aligned} G_0 = -1, G_1 = -1, G_2 = (-4)B_1, G_3 = 0, \\ G_4 = (-4)^2 B_2, G_5 = 0, G_6 = (-4)^3 B_3, \end{aligned}$$

and so on, the well-known scale of relation for Bernoulli's numbers may (provided only that n be odd) be written under the form

$$\sum_{k=0}^{n-1} (-)^k \beta^k n \cdot G_{n-k} = 0.$$

If in this formula we suppress the n which intervenes between the operative symbol β^k and G_{n-k} , so that the former is brought into juxtaposition with and acts on the latter, it becomes identical with that which we have found for the sharpened Hamiltonian numbers.

Those who wish to pursue the subject further may consult my memoir "On the so-called Tschirnhausen Transformation" (*Crelle*, vol. c. pp. 465-86), another "On Hamilton's Numbers," by Mr. Hammond and myself conjointly, just published in the Philosophical Transactions, and an addition thereto about to be presented to the Royal Society, in which a large generalization of the theory discussed by Hamilton in his Report to the British Association for 1836, but not brought by him to perfection, is resolved with a completeness which leaves nothing to be desired.

J. J. SYLVESTER.

New College, Oxford, October 1.