yellow, orange, red. A little later, the same night, on the brow of the ridge, in the faint mist which rose in masses, the bow was again seen vividly, against a background of trees; the bow being within 40 paces of the observer. W. G. BROWN.

Washington and Lee University, Lexington, Va.
September 23.

Destruction of Young Fish.

MAY I call your attention to the wholesale destruction of young fish which is carried on to a great extent round our coast? A few facts may not be out of place. Recently I have been visiting a small fishing village on the east coast, and have carefully noted the amount of young fish rejected by the fishermen on their return from trawling and shrimping. For example, from $4\frac{1}{4}$ pecks of shrimps no less than 793 flat-fish (dabs, soles, and turbots) were thrown on the beach useless; to this must be added about 200 whiting and an amount of young cod, herring, and skate beyond my power to count. Surely something can be done to remedy this! It is well known to the fishermen that the net does not injure the fish; so that before landing, if the net was roughly examined, all young fish could be thrown into the water again. DAVID WILSON-BARKER.

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ON HAMILTON'S NUMBERS.

FOLLOWING in the footsteps of Hamilton in his Report to the British Association, contained in the Proceedings for the year 1836, we may arrive at a solution, in a certain sense the simplest, of a problem in algebra the origin of which reaches back to Tschirnhausen, born 1651, deceased 1708. Every tyro knows how a quadratic equation, and all equations of a superior degree thereto, may be transformed into another in which the second term is wanting. Tschirnhausen showed that a cubic equation, and all equations superior in degree to the cubic, might be deprived of their second and third terms by solving linear and quadratic equations. Then over a century later Bring, of the University of Lund, in 1786 showed that every equation of the 5th, or any higher degree, might be deprived of its first three terms by means of solving certain cubic, quadratic, and linear equations.[1] What, then, it may be asked, is the law of the progression of which the three first terms are 2, 3, 5? What is the lowest degree an equation can have in order that it may admit of being deprived of four consecutive terms by aid of equations of the 1st, 2nd, 3rd, and 4th degrees, or more generally of i consecutive terms by aid of equations of the 1st, 2nd, 3rd, . . . and ith degrees, *i.e.* by equations none of a higher degree than the ith?[2]

In the 100th volume of *Crelle's Journal* (1886) I have shown that the progression continued as far as the case of eight terms being abolished is as follows—

2, 3, 5, 10, 44, 905, 409181, 83762797734.

These, with the exception of the three first, are not exactly what I call Hamilton's numbers, but serve to lead up to them.

Hamilton's numbers are—

2, 3, 5, 11, 47, 923, 409619, 83763206255, . . .

I will endeavour to explain wherein the difference consists between the two series.

Whilst it is true that four terms may be abolished in an equation of the 10th degree without solving equations beyond the 4th degree, there is this difference in favour of equations of the 11th or any higher degree, viz. that *fewer* biquadratics will be required for them than in the case of an equation limited to the 10th degree. And so in general whether we take, as our inferior limit to the degree of the equation to be transformed, the ith number in the upper series or the ith number in the lower one—whilst in neither case it will be necessary to solve any equations of a degree exceeding i—the total system in the latter case will be of a simpler character than in the former.

The numbers which I have named in honour of Hamilton may be obtained by a process exhibited in the table below—

1	0	0	0	0	0	0	0	...
	1	1	1	1	1	1	1	...
		2	3	4	5	6	7	...
		1	5	9	14	20	27	...
			6	15	29	49	76	...
			5	21	50	99	175	...
			4	26	76	175	350	...
			3	30	106	281	631	...
			2	33	139	420	1051	...
			1	35	174	594	1645	...
				36	210	804	2449	...
				&c.	&c.	&c.	&c.	...

We may now isolate the greatest figure which occurs in each column, and in this may we obtain the numbers 1, 1, 2, 6, 36 . . . which I call hypotenusal numbers; then adding these numbers together and increasing each sum so obtained by unity we arrive at the so-called Hamilton's numbers, viz. 2, 3, 5, 11, 47. . . . Now the question arises as to how they may be calculated; for obviously the crude method above given will be impossible to carry out in practice beyond the first few numbers in the scale. The method of generating functions—of which the idea occurred first to my coadjutor Mr. James Hammond, which certainly ought not to, and probably in the long run could not, have escaped me—leads to a wonderfully beautiful law, by means of which these numbers may be derived successively each from those that go before, just as is the case with Bernouilli's numbers.

The simplest and best mode of proceeding is as

[1] In a letter to Leibnitz (1677), which I have not seen, and the *Acta Eruditorum* for 1683.

[2] How the elevation of the degree of the equation to be transformed makes it possible to abolish a greater number (μ) of terms by an auxiliary system of equations of degrees none exceeding μ will be understood if we consider the cases of a quintic and quartic.

Supposing $(x, 1)^5$ to be a given quintic, on writing

$$\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon = 0$$

we obtain, by elimination of x, $(\alpha, \beta, \gamma, \delta, \epsilon)^5 = 0$, and any solution of his equation will enable us, by a well-known process, to find x by a linear equation.

If we select any letter, α, of the five we may equate it to a linear function of $y, \beta, \gamma, \delta, \epsilon$, so as to obtain

$$y^5 + (\beta, \gamma, \delta, \epsilon)^2 y^3 + (\beta, \gamma, \delta, \epsilon)^3 y^2 + (\beta, \gamma, \delta, \epsilon)^4 y + (\beta, \gamma, \delta, \epsilon)^5 = 0.$$

If in this equation we can find any system of ratios $\beta : \gamma : \delta : \epsilon$ such that $(\beta, \gamma, \delta, \epsilon)^2 = 0$, and $(\beta, \gamma, \delta, \epsilon)^3 = 0$, we can find y by solving a trinomial quintic, and therefore a system of admissible ratios $\alpha : \beta : \gamma : \delta : \epsilon$ becomes known.

All that is requisite therefore is to be able to obtain any point whatever of intersection of two given quadratic and cubic surfaces represented by $(\beta, \gamma, \delta, \epsilon)^2$ and $(\beta, \gamma, \delta, \epsilon)^3$ which obviously may be done by first finding a point (any point) in the quadratic surface (which only necessitates solving some quadratic equation or other); second, at this point drawing a right line (either one of a pair) lying on the surface, which may be effected by a well-known method involving only the solution of a quadratic; and third, finding any one of the three intersections of such line with the cubic surface.

Thus, then, by solving quadratic and cubic equations a quintic may be deprived of three consecutive terms. But not so a quartic; for in the case of a quartic we could not (with any real advantage) use a subsidiary equation of a higher degree than the 3rd, we should thus have only three letters, β, γ, δ, instead of four in the equation in y, and to make $(\beta, \gamma, \delta)^2 = 0$, $(\beta, \gamma, \delta)^3 = 0$, simultaneously, is the problem of finding an intersection of a quadratic and a cubic curve, which necessitates the solution of an equation of the 6th degree.

In the case of the quintic it may be well to notice that the ratios $\alpha : \beta : \gamma : \delta : \epsilon$ will not be all real, and consequently the trinomial quintic into which the original one has been transformed will not have its coefficients real, unless the quadric surface is a hyperboloid of one sheet (since it is only that species of quadric surfaces which contains real straight lines); and I have shown in my paper in Crelle that this is the case then, and then only, when the original quintic has four imaginary roots.

[2] If the number of equations of degrees $i, i-1, i-2, \ldots$ to be solved in the one case reckoned in DESCENDING order are $a, b, c, \ldots l, \ldots$ and in the other $a', b', c', \ldots l' \ldots$ respectively, if l, l' are the two first corresponding numbers which are not identical l' will be less than l.