Supplementary methods 1: Decomposition of the Hopf fields into vector spherical harmonics

We provide some intermediate steps in the calculation of the vector spherical harmonic coefficients (VSPH) for the electromagnetic Hopf fields. The vector potential $\boldsymbol{A}_{lm}(k,\boldsymbol{r})$, for the VSPHs is:

$$\boldsymbol{A}^{TE}_{lm}(k,\boldsymbol{r}) = \frac{1}{i\omega} f_l(kr)\boldsymbol{L}Y_{lm}(\theta,\phi) \quad (1)$$

$$\boldsymbol{A}^{TM}_{lm}(k,\boldsymbol{r}) = \frac{1}{k^2}\boldsymbol{\nabla}\times\Big[f_l(kr)\boldsymbol{L}Y_{lm}(\theta,\phi)\Big] \quad (2)$$

where $\boldsymbol{L} = -i\boldsymbol{r}\times\boldsymbol{\nabla}$ and $f_l(kr)$ is a linear combination of the spherical bessel functions $j_l(kr)$, $n_l(kr)$, determined by boundary conditions.

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{1+2l}{2\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}L^m_l(\cos(\theta))e^{im\phi}$$

where L^m_l are associated Legendre poynomials. In free-space $f_l(kr) = j_l(kr)/\sqrt{l(l+1)}$,

$$j_l(z) = \sqrt{\frac{\pi}{2z}}J_{l+\frac{1}{2}}(z)$$

where J_l are Bessel functions of the first kind. A general free-space vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ can be expressed in the spherical harmonic basis as:

$$\boldsymbol{A}(\boldsymbol{r},t) = \int \mathrm{d}k\sum_{l=1}^{\infty}\sum_{m=-l}^{l}[\alpha^{TM}_{lm}(k)\boldsymbol{A}^{TM}_{lm}(k,\boldsymbol{r}) + \alpha^{TE}_{lm}(k)\boldsymbol{A}^{TE}_{lm}(k,\boldsymbol{r})]e^{-i\omega t} + c.c. \quad (3)$$

To determine the coefficients $\alpha^{TE/TM}_{lm}(k)$ we first compute the Fourier transform of the field at time $t=0$:

$$\boldsymbol{B}(\boldsymbol{r},t=0) = \frac{1}{\pi(r^2+1)^3}\big(1+x^2-y^2-z^2, 2(xy-z), 2(y+xz)\big) \quad (4)$$

$$\boldsymbol{E}(\boldsymbol{r},t=0) = \frac{2}{\pi(r^2+1)^3}\big(2(xy+z), 1+y^2-x^2-z^2, -2(x-yz)\big) \quad (5)$$

Noting that: $\int \mathrm{d}^3x f(\boldsymbol{x}) x_j e^{\pm i\boldsymbol{k}\cdot\boldsymbol{x}} = (\mp i\partial_{k_j}) \int \mathrm{d}^3x f(x) e^{\pm i\boldsymbol{k}\cdot\boldsymbol{x}}$ and using the following transform: $\int \mathrm{d}^3x 1/(a^2+r^2-d)^{n+1} e^{\pm i\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{\partial_d^n}{n!}\frac{2\pi^2}{k} e^{-k\sqrt{a^2-d}}|_{d=0}$, we obtain the following transforms:

$$
\begin{aligned}
\tilde{\boldsymbol{B}}(\boldsymbol{k}) &= \pi\frac{e^{-k}}{k}\left(k_y^2+k_z^2, ikk_z-k_xk_y, -ikk_y-k_xk_z\right) &(6)\\
\tilde{\boldsymbol{E}}(\boldsymbol{k}) &= \pi\frac{e^{-k}}{k}\left(-ikk_z-k_xk_y, k_x^2+k_z^2, ikk_x-k_yk_z\right) &(7)
\end{aligned}
$$

with $\boldsymbol{E},\boldsymbol{B}(\boldsymbol{r}) = \int \mathrm{d}^3k \tilde{\boldsymbol{E}},\tilde{\boldsymbol{B}}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. Further noting that $\int \mathrm{d}^3k f(\boldsymbol{k}) k_j e^{\pm i\boldsymbol{k}\cdot\boldsymbol{x}} = (\mp i\partial_{x_j}) \int \mathrm{d}^3k f(k) e^{\pm i\boldsymbol{k}\cdot\boldsymbol{x}}$, and that $\int \mathrm{d}^3k f(k) e^{\pm i\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{4\pi}{r}\int \mathrm{d}k f(k) k \sin(kr)$, the electric and magnetic fields at $t=0$ can be expressed as:

$$
\begin{aligned}
\boldsymbol{B}(\boldsymbol{r}) &= 4\pi^2\int \mathrm{d}k e^{-k}\left(-\partial_y^2-\partial_z^2, +\partial_x\partial_y+k\partial_z, \partial_x\partial_z-k\partial_y\right)\frac{\sin(kr)}{r} &(8)\\
\boldsymbol{E}(\boldsymbol{r}) &= 4\pi^2\int \mathrm{d}k e^{-k}\left(\partial_x\partial_y-k\partial_z, -\partial_x^2-\partial_z^2, k\partial_x+\partial_y\partial_z\right)\frac{\sin(kr)}{r} &(9)
\end{aligned}
$$

from which the coefficients can be derived for each frequency component separately through: $\alpha_{lm}^{TE}(k) j_l(kr) = k/\sqrt{l(l+1)}\int \mathrm{d}\Omega Y_{l,m}^* \boldsymbol{B}\cdot\boldsymbol{r}$ and $\alpha_{lm}^{TM}(k) j_l(kr) = -k/\sqrt{l(l+1)}\int \mathrm{d}\Omega Y_{l,m}^* \boldsymbol{E}\cdot\boldsymbol{r}$ to obtain:

$$
\begin{aligned}
\alpha_{lm}^{TE}(k) &= \begin{cases} \sqrt{\frac{4\pi}{3}}k^3e^{-k} & \text{for} \quad l=m=1 \\ 0 & \text{for} \quad l\neq 1, m\neq 1 \end{cases} &(10)\\
\alpha_{lm}^{TM}(k) &= -i\alpha_{lm}^{TE}(k) &(11)
\end{aligned}
$$

The vector potential for the Hopf knots can therefore be expressed as:

$$
\boldsymbol{A}_{\mathrm{Hopf}}(\boldsymbol{r},t) = \sqrt{\frac{4}{3\pi}}\int \mathrm{d}k\, k^3 e^{-k}\left[\boldsymbol{A}_{1,1}^{TE}(k,\boldsymbol{r}) - i\boldsymbol{A}_{1,1}^{TM}(k,\boldsymbol{r})\right]e^{-i\omega t} + c.c., \qquad (12)
$$

Time evolution of the Hopf field lines: Magnetic field lines

Quicktime movie file (620 KB): Supplementary video 1.mov

Summary: Animation showing the time evolution of the Hopf Magnetic field lines, some time steps of which are shown in figure 2.

Time evolution of the Hopf field lines: Electric field lines

Quicktime movie file (772 KB): Supplementary video 2.mov

Summary: Animation showing the time evolution of the Hopf Electric field lines, some time steps of which are shown in figure 2.

Time evolution of the Hopf field lines: Electric and Magnetic field lines

Quicktime movie file (852 KB): Supplementary video 3.mov

Summary: Animation showing the time evolution of the Hopf Electric and Magnetic field lines, some time steps of which are shown in figure 2.