# The Differential Analyser

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## PURPOSE

THE application of mathematics to problems both of pure and applied science often leads to differential equations which have no formal solution in quadratures or in terms of tabulated functions, but for which numerical values of the solutions are required. Until recently, the only available methods for evaluating the solutions of such equations were graphical methods, which are rather limited in scope and accuracy, and numerical methods, which are lengthy and require continual concentrated attention on the part of the worker, and rapidly become more laborious the more elaborate the equations. So the development of a mechanical method, rapid, accurate, and applicable to a wide range of equations, is an advance of considerable importance, with applications to a wide range of problems of scientific and technical interest.

Such an advance has been made by Dr. V. Bush, of the Massachusetts Institute of Technology, by the development of a machine known as the differential analyser, of which the first was designed and built there[1]. The general idea of such a machine in the abstract is due to Lord Kelvin[2], but the practical design of a machine which could be made, and would work accurately when made, is due essentially to Dr. Bush.

A similar machine, with which the writer has been closely concerned, has been built at the University of Manchester; a short notice of the formal opening of this machine appeared in NATURE recently[3]. Another such machine has quite recently been built at the University of Pennsylvania.

## CONSTRUCTION

The machine consists of a number of units which can be connected to shafts which drive them or are driven by them. These shafts can be connected together by gearing in various ways, so that the relations between the rotations of the different shafts satisfy various differential equations. The adaptability of the machine as a whole, which is one of its most important features, depends essentially on the wide range of possibilities of such interconnexions.

The essential units are those called integrators, since they carry out mechanically the operation of integration. Each of these is a continuously variable gear, consisting of a friction drive from a horizontal disc, which can rotate about a vertical axis, to a vertical wheel resting on it; the distance from the centre of the disc to the point of contact of the wheel with it can be varied by displacing the disc, the axle of which is carried in bearings in a carriage which can move along a pair of guide bars. This displacement of the disc represents the integrand, the rotation of the disc represents the variable of integration, and the rotation of the wheel represents the result of the integration (Fig. 1).

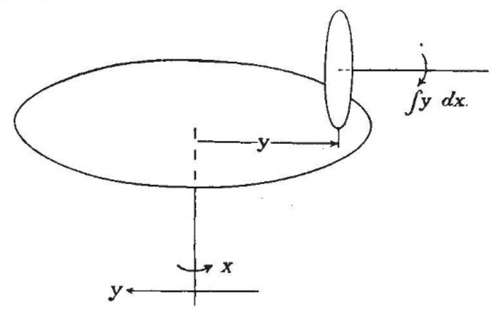


FIG. 1. Principle of the integrator.

It is essential for accurate operation of the machine that, in the rotational motion of the disc, there should be no slip between disc and wheel at the point of contact. On the other hand, it is often necessary to make such connexions that a considerable amount of mechanism is to be driven by the rotation of the wheel, and the friction at the point of contact between disc and wheel is quite insufficient to provide the necessary torque on the shaft. To avoid this difficulty, an ingenious mechanical servo-mechanism, called a torque-amplifier, has been developed by Dr. Bush for this purpose. Its operation is similar in principle to a power-operated capstan. A band passes round a drum which is driven by an independent source of power and is continually running; one end of the band is fixed to an arm on the shaft carrying the integrating wheel, and the other to an arm on another shaft (output shaft) coaxial with it. The rotation of the integrating wheel tightens the band and therefore increases the friction between it and the drum, and this additional frictional force on the band pulls round the arm attached to the output shaft, so that in effect the rotation of the integrating wheel simply operates as a control of the supply of power to the output shaft. As developed by Dr. Bush, this control is very delicate, the torque required to operate it being about one ten-thousandth of the torque required to drive the output shaft, and this small torque can be provided by friction between disc and integrating wheel without any danger of slip.

Other units are input tables, used for supplying to the machine information in the form of a functional relation between variables occurring in the equation. A graph representing this relation is fixed to a board spanned by a movable bridge carrying a pointer which can be moved along it. The position of this pointer along the bridge is controlled by rotation of a handle, which also drives the shaft to which the information expressed by the curve is to be transmitted. The bridge is moved across the table by the operation of the machine, and an operator stationed at the table turns the handle to keep the pointer on the curve as the bridge moves.

There is also an output table, similar in construction to an input table, on which the machine delivers the result in the form of a graph of the solution of the equation. For a general survey of the behaviour of the solution, this is the most convenient form of record, but it is not suitable if quantitative results are required, on account of the distortion of paper with changes of temperature, etc., and also on account of the time required to measure up the curves. An alternative recorder is a camera which takes photographs, at selected intervals, of a set of revolution counters which can be connected to the appropriate shafts.

In addition, there are differential gears which serve to add or subtract the rotations of two shafts, and 'front-lash units', the object of which is to compensate any backlash in the various drives. This is achieved by a gear train arranged to give a small and adjustable angular advance of a driven shaft relative to a driving shaft, at each occasion on which the direction of rotation of the latter changes.

The machine is driven by electric motors. One motor drives the shaft the rotation of which represents the independent variable, and each pair of integrators has a motor driving their two torque amplifiers, one for each integrator. The running of the machine is controlled by a set of switches placed conveniently for an operator who may be following a curve on an input table. These switches operate contactors which control the supply of current to the various motors.

A general view of the machine at the University of Manchester is shown in Fig. 2, in which the main component units are indicated.

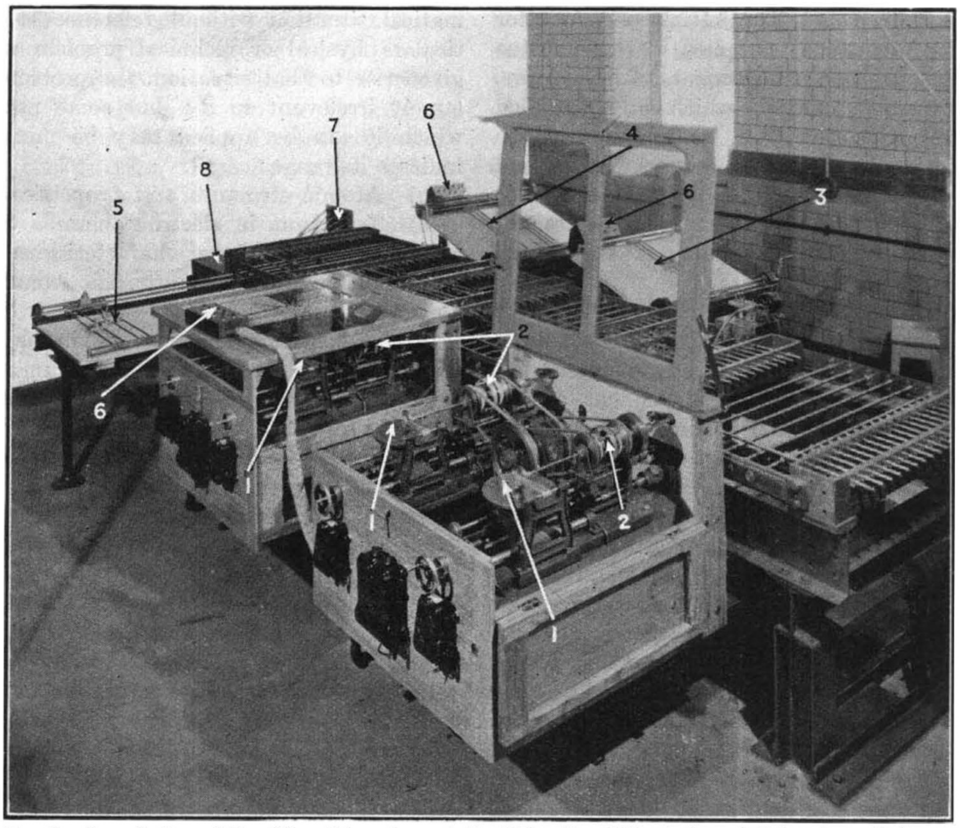


FIG. 2. General view of the differential analyser at the University of Manchester. 1, Integrators; 2, torque amplifiers; 3, input table; 4, special input table; 5, output table; 6, control switches; 7, revolution counters; 8, base for camera (camera not shown). Photograph by Metropolitan-Vickers Electrical Co., Ltd.

### OPERATION

The operation of the machine involves two processes: first that of setting up the interconnexions between the units in such a way that the relations between certain shafts satisfy the equation in question; and secondly the carrying out of an actual solution of the equation.

The setting-up is first carried out in diagrammatic form, on paper, and the scales on which the rotations of shafts represent variables in the equation are decided. At this stage, experience counts for a lot, in discovering how to put an equation on the machine, and in deciding between various alternative ways which may be possible. The interconnexions on the machine are then set up from the diagram. The initial conditions for the solution are usually supplied to the machine in the form of initial displacements of integrators.

There are many equations which can be handled without any attention from an operator in the course of the running of the machine, beyond pressing the control switches. Other equations require the use of one or more input tables, for each of which an operator is required as explained above.

The process of setting up the machine may take from half a day to a day, but once it is set up, a solution may be obtained in ten to fifteen minutes, this time being practically independent of the complexity of the equation so long as it is within range of the machine, whereas the process of solution of a comparatively simple equation by purely numerical means would probably take an experienced worker four hours to a day of work which needs continuous careful attention, as, although it is of a routine nature, it is not altogether simple and straightforward; a more elaborate equation, or a less experienced worker, would require much longer.

Use of the machine may thus not lead to a great saving of time when only a single solution of an equation is required, though even then there is considerable saving of mental labour of a routine kind. But when a number of solutions of one equation, for example, solutions with different initial conditions, or different values of numerical coefficients occurring in it, are required, the time occupied in setting up the machine for the equation is a small proportion of the whole period concerned with work on that equation, and the total time required to obtain a specified set of solutions may then be reduced by a factor of 10 or 20. This very large saving of time, and the corresponding saving of mental effort, is important not only in dealing with problems which would in time be dealt with in any case, but even more in making it practicable to undertake extended investigations which without such mechanical assistance would be altogether too laborious and time-consuming.

Further, since the time and labour of mechanical solution does not increase appreciably with the complexity of the equations, whereas the time and labour of numerical solution does increase very considerably, the value of the machine increases rapidly with the complexity and range of equations within its capacity, and this increases rapidly with the number of units of which it is constituted. A machine with eight integrators, for example, is very much more than twice as valuable as one with four integrators.

### Applications

Since the differential analyser handles a mathematical situation without reference to the particular physical or technical problem which has given rise to that situation, the problem itself is largely irrelevant to it; but some problems to which it can be applied may be mentioned to indicate its range:

(*a*) Atomic structure and properties.

(*b*) Transients in electrical circuits containing elements with non-linear characteristics.

(*c*) Performance of automatic control mechanism.

(*d*) Propagation of radio waves in the Heaviside layer, regarded as a stratified medium.

(*e*) Vibrations of systems with non-linear restoring forces.

(*f*) Paths of electrified particles in the field of a magnet (for example, in connexion with the theory of the aurora and of cosmic radiation).

(*g*) Equilibrium and stability of stellar structures.

By the nature of the case, the differential analyser provides a numerical (or graphical) solution of an equation with definite numerical values of coefficients in the equation, and of initial conditions, and cannot provide a general analytical solution; indeed, one of its virtues is that it can be applied to equations for which no such solution exists. Thus it does not seem likely to be of any great interest or value in purely mathematical fields, in which the interest, if any, in the solution of any equation lies in the general form of the solution, or in its formal analytical expression. On the other hand, in physical and technical investigations, it is often the special solutions, and actual numerical values, which are required; and it may happen that even if there is a formal analytical solution, it is unsuitable for numerical evaluation. So that it is mainly in connexion with such applications that the differential analyser is likely to be used.

### The Differential Analyser at the University of Manchester

In general design and in many details, the differential analyser at the University of Manchester, which has been constructed by the

Metropolitan-Vickers Electrical Co., Ltd., follows closely Dr. Bush's original machine at the Massachusetts Institute of Technology, but several modifications in detail have been made.

The machine is being built in two sections, one of which is now complete. This completed section comprises four integrators and two input tables, the output table, and a special camera designed and built by Messrs. Newman and Guardia, Ltd., for the photographic recorder. One of the input tables is of special construction, for handling equations describing the behaviour of a system in which the rate of change of a quantity at time $t$ may depend explicitly on its value, or on the values of other quantities, at time $t - T$ (where the time-lag $T$ may be constant or variable) as well as on their values at time $t$. The second section, now under construction, will comprise four further integrators, making eight altogether, and probably four further input tables, and, as explained above, the greater range will very greatly increase the value of the machine.

The construction of this differential analyser has been made possible, first through the great generosity of Mr. Robert McDougall, deputy treasurer of the University, who first gave to the University a donation to cover the estimated cost of the first section, now completed, and has recently supplemented this by a further donation to cover the completion of the machine in accordance with the original estimates; secondly, by the friendly and generous co-operation of Dr. Bush himself, who freely gave his drawings, and several suggestions for improvement, and helped greatly by his advice based on experience of construction and operation of his own machine; and thirdly, through the co-operation of Mr. A. P. M. Fleming, of Metropolitan-Vickers Electrical Co., Ltd., who undertook the construction of the machine, and of those members of the firm who have been concerned in its design, construction and erection.

[1] V. Bush, *J. Franklin Inst.*, Oct. 1931, and "Proc. Internat. Congress on Applied Mechanics", Cambridge, 1934.
[2] Sir W. Thomson, *Proc. Roy. Soc.*, **24**, 269; 1876.
NATURE, **135**, 535, April 6, 1935.

## Royal Society Discussion on Supraconductivity

In opening the Royal Society discussion on supraconductivity, Prof. J. C. McLennan referred first to new methods of helium liquefaction which have recently been developed. In Prof. F. Simon's method, high pressure helium gas is cooled to liquid hydrogen temperatures and then allowed to expand through a valve, a small quantity of liquid helium being produced which is quite adequate for many types of experiments. A second method, developed by Prof. P. Kapitza at Cambridge, applies to helium the method first used by Claude for the liquefaction of air, part of the gas doing external work in an expansion engine and so cooling the remainder below the inversion temperature of the Joule-Thomson effect, whence it can be liquefied by expansion.

Prof. McLennan then discussed the rapid progress in the attainment of temperatures near the absolute zero made possible by the adiabatic demagnetisation method, the latest experiments of de Haas with potassium chromium alum having reached a temperature of 0·0044° on the scale obtained by measuring the magnetisation and using an extrapolated Curie Law to obtain the temperature. Progress is at the same time being made in the establishment of the thermodynamic scale in this temperature region. Prof. W. H. Keesom reported that the thermodynamic scale has been established down to 0·9° K. using the helium thermometer. Prof. Simon reported the results of an ingenious method developed at Oxford in which the heat, $dQ$, required to warm up the salt from its lowest temperatures to its initial temperature is determined by using $\gamma$-rays to warm up the salt. Since the entropy change, $dS$, on magnetisation can be calculated and the cooling is adiabatic, the temperature on the absolute scale is obtained from $dQ/dS$. The results show that the thermodynamic scale and the magnetic scale using iron ammonium sulphate do not differ by more than 10 per cent down to 0·08° K., but that below this temperature the thermodynamic temperature is greater than the magnetic temperature. For these lower temperatures the shape of the specimen has a large effect on the magnetic temperature owing to the increasing importance of the demagnetising coefficient.

Prof. McLennan referred also to the prediction that at these low temperatures the effect of nuclear magnetic moments should become important, owing to the thermal energy becoming comparable with the energy of magnetisation due to nuclear moments. Prof. Simon considers that it may be possible to use a two-stage demagnetisation process for the attainment of the lowest temperatures, the nuclear moments becoming effective in the second stage. Dr. Heitler's prediction in the discussion that it would take a year for equilibrium to be set up owing to the smallness of the nuclear interaction failed to shake Prof. Simon's determination to try the experiment.

The production of these low temperatures has