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## General existence of locally distinguishable maximally entangled states only with two-way classical communication

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It is known that there exist two locally operational settings, local operations with one-way and two-way classical communication. And recently, some sets of maximally entangled states have been built in specific dimensional quantum systems, which can be locally distinguished only with two-way classical communication. In this paper, we show the existence of such sets is general, through constructing such sets in all the remaining quantum systems. Specifically, such sets including  $p$  or  $n$  maximally entangled states will be built in the quantum system of  $(np - 1) \otimes (np - 1)$  with  $n \geq 3$  and  $p$  being a prime number, which completes the picture that such sets do exist in every possible dimensional quantum system.

Local operations and classical communication (LOCC), as an important operational setting, has been widely considered in the discrimination of quantum states, which has played a significant role in understanding of the relationship between quantum entanglement and quantum nonlocality. On the one hand, quantum entanglement is not necessary for quantum nonlocality<sup>1–9</sup>. Bennett *et al.* firstly discovered a locally indistinguishable  $3 \otimes 3$  pure product basis<sup>1</sup>, which has been generalized to higher dimensional quantum systems<sup>2–6</sup>, revealing the phenomenon of “nonlocality without entanglement” to a greater extent. As another interesting finding, there exist orthogonal states whose local indistinguishability is increased with less entanglement<sup>10</sup>. On the other hand, quantum entanglement is not sufficient for quantum nonlocality<sup>11–19</sup>. Any two orthogonal quantum states, entangled or not, can be perfectly distinguished locally<sup>11</sup>. And for maximally entangled states (MESs), Nathanson has proved any three orthogonal MESs in a  $3 \otimes 3$  quantum system can be distinguished by LOCC<sup>12</sup>. Then Fan has showed that any  $l$  mutually orthogonal MESs, which are in canonical forms, are locally distinguishable if  $l(l - 1) \leq 2p$  with  $p$  being a prime in a  $p \otimes p$  quantum system<sup>13</sup>, which has been extended to prime power quantum system<sup>14</sup>.

According to the variance of classical communication, there exist two kinds of LOCC settings. We denote local operations and one-way classical communication by “1-LOCC”, and similarly, “2-LOCC” represents local operations and two-way classical communication. Actually, the 2-LOCC is LOCC in common sense, and we will use the expression 2-LOCC to show the difference with 1-LOCC clearly in the following parts. Because of the complexity of 2-LOCC operations, people usually apply either the super sets or the sub sets to study the local discrimination of quantum states. Taking use of the super sets, positive partial transpose (PPT) operations, Yu *et al.* have come up with four locally indistinguishable ququad-ququad MESs<sup>20</sup>, which has been expanded in two power dimensional quantum system<sup>21,22</sup>. On the contrary, the researching progresses<sup>11–14</sup> have been made mostly based on its subset, 1-LOCC. That is, the local distinguishability of a set is ensured by its 1-LOCC distinguishability. Thus a natural question is whether there exist some quantum states sets or not, which can be distinguished by 2-LOCC but not by 1-LOCC. The existence of such sets, we call 2-LOCC sets, can help to show the difference between 1-LOCC and 2-LOCC, i.e., two-way classical communication has advantage over one-way classical communication in fact. Obviously, the cardinalities of the 2-LOCC sets are small compared to the dimension of their corresponding quantum system, because no  $k > d$  MESs can be locally distinguished<sup>12,20,21</sup>. In ref. 15, the author has given two triple 2-LOCC sets in the quantum system of dimension  $2m \otimes 2m$  and  $3r + 2$ . Lately, we have constructed 2-LOCC sets including four or five MESs in  $4m \otimes 4m$  system, and similarly,  $3R$  MESs in  $2Rm \otimes 2Rm$  system,  $4R$  and  $5R$  MESs in  $4Rm \otimes 4Rm$  system, where  $R = 2^r$  with  $r$  being a positive integer<sup>16</sup>.

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However, all the existing results do not cover every possible dimensional quantum system, that is, it is still unknown whether the existence of 2-LOCC sets is general or restricted to specific dimensions. In this paper, we answer this question in positive. We start with presenting a decomposition of all the dimension numbers  $\mathbb{D} = \mathbb{Z}_+ / \{1, 2, 3\} = \{np - 1, n \geq 3\} \cup \{p - 1\} / \{1, 2\}$ , where  $p$  is a prime number. Combined with the previous result that there exists a 2-LOCC set in even dimensional quantum system, we only need to build 2-LOCC sets in the remaining dimensions  $\{np - 1, n \geq 3\}$  because there is no 2-LOCC sets in  $3 \otimes 3$  system. The detailed form,  $np - 1, n \geq 3$ , motivates us to consider using the second 2-LOCC distinguishing protocol in ref. 15 to complete the picture. Before constructing more 2-LOCC sets, we should make a simplification of the referred 2-LOCC distinguishing protocol. Specifically, Bob's local measurement elements become less, which makes the following construction more efficient. Next, we build  $a + 1$  mutually orthogonal MESs in  $d \otimes d$  quantum system with  $d = a + (a + 1)t$  base on the simplified 2-LOCC distinguishing protocol. Finally, because of  $np - 1 = p(n - 1) + (p - 1) = n(p - 1) + (n - 1), n \geq 3$ , we can construct  $p$  or  $n$  MESs as 2-LOCC sets easily in the remaining quantum systems, which ensures the general existence of 2-LOCC sets in every possible dimensional quantum system.

### Results

**Notations.** Consider a bipartite quantum system  $A \otimes B$  of the dimension  $d \otimes d$ , that is, either subsystem  $A$  or  $B$  can be regarded as a qudit with  $d$  levels labeled by  $\mathbb{Z}_d = \{0, 1, \dots, d - 1\}$ . Let  $\{|a\rangle | a \in \mathbb{Z}_d\}$  be the computational basis, and the standard bipartite maximally entangled state in this system is expressed as  $|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{j \in \mathbb{Z}_d} |jj\rangle$ . We can also define the bit flip and phase flip operators to be

$$\mathcal{X}_d = \sum_{a \in \mathbb{Z}_d} |a \oplus_d 1\rangle \langle a|, \quad \mathcal{Z}_d = \sum_{a \in \mathbb{Z}_d} \omega_d^a |a\rangle \langle a| \tag{1}$$

where  $\omega_d = e^{\frac{2\pi i}{d}}$  and the subscript denotes the dimension of the corresponding quantum system. When the dimension is 2, the above two operators will become the Pauli matrices  $X$  and  $Z$ , respectively. Thus the matrices

$$U_{ts} = \mathcal{X}_d^s \mathcal{Z}_d^t \tag{2}$$

generated by  $\mathcal{X}_d$  and  $\mathcal{Z}_d$  in Eq. (1) have been called generalized Pauli matrices<sup>13</sup>, GPMs for short. Because of the one-to-one relationship between MES and the corresponding unitary operation, we denote the set  $\{|\Psi_i\rangle = (I \otimes U_i)|\Phi\rangle\}_{i=0}^{n-1}$  by the corresponding unitary set as  $\{U_i\}_{i=0}^{n-1}$ , where  $|\Phi\rangle$  is the standard MES.

There has already been a necessary and sufficient condition<sup>15</sup>, which will be employed in the following part repeatedly.

**Lemma 1.** (*M. Nathanson*<sup>15</sup>) *Given a set of states  $S = \{|\Psi_i\rangle = (I \otimes U_i)|\Phi\rangle\} \subset \mathbb{C}^d \otimes \mathbb{C}^d$ , with  $|\Phi\rangle$  the standard maximally entangled state, the elements of  $S$  can be perfectly distinguished with one-way LOCC if and only if there exists a set of states  $\{|\phi_k\rangle\} \subset \mathbb{C}^d$  and a set of positive numbers  $\{m_k\}$  such that  $\sum_k m_k |\phi_k\rangle \langle \phi_k| = I_d$  and*

$$\langle \phi_k | U_j^\dagger U_i | \phi_k \rangle = 0$$

whenever  $i \neq j$ .

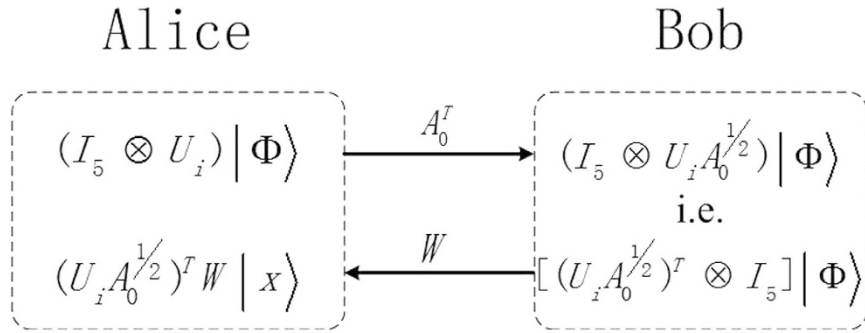
*Equivalently, the elements of  $S$  can be perfectly distinguished with one-way LOCC if and only if there exists a  $d \times r$  partial isometry  $W$  such that  $WW^\dagger = I_d$  and such that whenever  $i \neq j$ , the  $r \times r$  matrix  $W^\dagger U_j^\dagger U_i W$  has every diagonal entry equal to zero.*

**Decomposition of all the dimension numbers.** So far, it has already been proved in refs 15 and 16 that there exist 2-LOCC sets in the quantum system of  $2m \otimes 2m, (3r + 2) \otimes (3r + 2), 4m \otimes 4m, 2Rm \otimes 2Rm$  and  $4Rm \otimes 4Rm$ , where  $R = 2^r$  with  $r$  and  $m$  being a positive integer. But obviously, the union of  $\{2m\}, \{3r + 2\}, \{4m\}, \{2Rm\}$  and  $\{4Rm\}$  does not cover all the dimension numbers, or to say all the positive integers greater than one, which motivates us to generalize the result. That is, we want to look for some 2-LOCC set(s) in every-dimension quantum system. Before that, we should be clear about the mathematical structure of all the dimension numbers. Therefore, we will start with a decomposition of all the positive integers, denoted by  $\mathbb{Z}_+$ , as the following lemma, which may not be the best one but can satisfy our requirements.

**Lemma 2.** *The set of all dimension numbers, denoted by  $\mathbb{Z}_+$ , is equivalent to the union of  $\{np - 1, n \geq 3\}, \{3\}$  and  $\{p - 1\}$  with  $p$  being a prime number. That is to say,  $\mathbb{Z}_+ = \{np - 1, n \geq 3\} \cup \{3\} \cup \{p - 1\}$ .*

*Proof.* It is obvious that every number in the union  $\{np - 1, n \geq 3\} \cup \{3\} \cup \{p - 1\}$  is positive, i.e.,  $\{np - 1, n \geq 3\} \cup \{3\} \cup \{p - 1\} \subset \mathbb{Z}_+$ . Next, we will explain each positive number in  $\mathbb{Z}_+$  can be expressed as an element in the union.

As the fundamental theorem of arithmetic stated, every integer greater than one either is prime itself or is the product of prime numbers, and that this product is unique, up to the order of the factors. Thus we have  $\mathbb{Z}_+ = \{np - 1, n \geq 1\}$  with  $p$  being a prime, that is,  $\mathbb{Z}_+ = \{np - 1, n \geq 3\} \cup \{2p - 1\} \cup \{p - 1\}$ . Because the set  $\{(2p - 1)/\{3\}\} \subset \{np - 1, n \geq 3\}$  when the latter  $p$  equals to 2, we can replace  $\{2p - 1\}$  by  $\{3\}$  for simplicity. In addition, all the elements in the set  $\{p - 1\}$  cannot belong to  $\{np - 1, n \geq 3\} \cup \{3\}$ , thus we have the final decomposition  $\mathbb{Z}_+ = \{np - 1, n \geq 3\} \cup \{3\} \cup \{p - 1\}$ , which completes our proof.  $\square$



**Figure 1.** Alice measures with  $\{A_k^T\}$  locally and communicates the result  $0$  to Bob. Then Bob does the measurement  $\{W^T W\}$  on his own system and transmits the corresponding result to Alice. At this point, the states in Alice’s part are always mutually orthogonal whatever Bob’s result is, which ensures the 2-LOCC distinguishability.

For the dimension 2, any two orthogonal MESs can be discriminated by 1-LOCC<sup>11</sup>, while any three orthogonal MESs cannot be distinguished by 2-LOCC<sup>20</sup>. And for the dimension 3, any three orthogonal MESs are 1-LOCC distinguishable<sup>12</sup>, while any four orthogonal MESs are 2-LOCC indistinguishable<sup>20</sup>. Thus the dimensions of quantum system, denoted by  $\mathbb{D}$ , we consider to build 2-LOCC sets should be not smaller than 4, that is,

$$\mathbb{D} = \mathbb{Z}_+ / \{1, 2, 3\} = \{np - 1, n \geq 3\} \cup \{p - 1\} / \{1, 2\}.$$

And when the dimension number belongs to  $\{p - 1\}$ , which must be even, 2-LOCC sets have been presented in ref. 15. Therefore, our objective becomes to construct 2-LOCC sets in the quantum system of dimension numbers  $d \in \{np - 1, n \geq 3\}$ , where  $p$  is a prime number. We rewritten the form of “ $np - 1, n \geq 3$ ” to

$$np - 1 = p(n - 1) + (p - 1) = n(p - 1) + (n - 1), \quad n \geq 3,$$

which makes us associate the structure of the second kind of 2-LOCC sets in ref. 15 in the  $(3r + 2) \otimes (3r + 2)$  quantum system. However, to make full use of that 2-LOCC distinguishing protocol efficiently, we will give it a simpler presentation. That is, a simplified 2-LOCC protocol, we prefer omitting the word “distinguishing” for simplicity, will be discussed as in the following subsection.

**Simplified 2-LOCC protocol.** In ref. 15, the author has already proposed two triple MESs sets, which can be distinguished by 2-LOCC but not by 1-LOCC. The 1-LOCC indistinguishability of both sets has been assured by the above sufficient and necessary condition, while the 2-LOCC distinguishability has been obtained by two different protocols. We have successfully employed the first 2-LOCC protocol to construct more 2-LOCC sets in ref. 16. As for the second 2-LOCC protocol to distinguish the three MESs below, actually, it can be simplified further.

$$\{U_i\}_{i=0}^2 = \left\{ \begin{pmatrix} I_2 & \\ & \mathcal{I}_3 \end{pmatrix}, \begin{pmatrix} \omega_1 X & \\ & \mathcal{X}_3 \end{pmatrix}, \begin{pmatrix} \omega_2 Z & \\ & \mathcal{X}_3^2 \end{pmatrix} \right\}$$

We will present the detailed 2-LOCC distinguishing protocol, which is simpler in Bob’s choice of local measurement. First, Alice does the same measurement as is shown in ref. 15 with  $\{A_k^T\}$ , where

$$A_k = \frac{1}{3} \sum_{j=0}^2 |j\rangle \langle j + k + 2| \langle k + 2|.$$

If Alice’s outcome is  $k = 0$ , without loss of generality, then the original 3 local unitaries  $U_i$  will become

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{3}} I & \\ & |0\rangle \langle 0| \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \omega_1 X & \\ & |1\rangle \langle 0| \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \omega_2 Z & \\ & |2\rangle \langle 0| \end{pmatrix} \right\}.$$

Next, we will build a simpler  $W$ , in which each column acts as Bob’s measurement element,

$$W = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & -\sqrt{3} & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \\ -\omega_2 & -\omega_2 & -\omega_2 & -\omega_2 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

which satisfies the condition that  $W^\dagger (U_j^T)^\dagger A_k U_i^T W$  has diagonal element equal to zero. After Bob transforming this measurement result to Alice, she can finish the local discrimination finally. The whole distinguishing process can be described in the following Fig. 1.

The above 2-LOCC protocol has been much simplified compared to the one given in the ref. 15, because we need only one basis to form the rank-1 measurement operators. That is, we do not need that many measurement elements  $(u, v)$  to distinguish the three MESs using local operations and two-way classical communication.

**Constructing more 2-LOCC sets.** Taking use of the above simplified 2-LOCC protocol, we will directly build more 2-LOCC sets of orthogonal MESs in this subsection, which can help us to generalize the existence of 2-LOCC in every possible dimensional quantum system in the next subsection.

Consider the following  $a + 1$  mutually orthogonal MESs  $\{U_i|i = 0, 1, \dots, a\}$  in  $d \otimes d$  quantum system with  $d = a + (a + 1)r$  and  $r$  being an integer, where

$$U_i = \begin{pmatrix} \delta_i S_i & \\ & P_i \otimes \mathcal{I}_r \end{pmatrix}$$

with

$$\begin{cases} S_i = \begin{cases} \mathcal{X}_a^i, & i = 0, 1, \dots, a - 1, \\ \mathcal{Z}_a, & i = a. \end{cases} \\ P_i = \mathcal{X}_{a+1}^i, & i = 0, 1, \dots, a. \end{cases}$$

as in Eq. (1), but the dimension is  $a, a + 1$  respectively. And we should limit the coefficients to satisfy

$$\begin{cases} |\delta_i| = 1, & i = 0, 1, \dots, a; \\ \overline{\delta_0} \delta_j^2 \overline{\delta_{2j}} \neq 1, & j = 1, \dots, \left\lfloor \frac{a-1}{2} \right\rfloor, \text{ if } a \text{ is odd, these two limitations will be enough.} \\ \overline{\delta_0} \delta_{\frac{a}{2}} \delta_{\frac{a-1}{2}} \overline{\delta_{a-1}} \neq 1, & \text{ if } a \text{ is even, we also need this additional condition.} \end{cases} \quad (3)$$

Then we have Theorem as follows.

**Theorem 1.** *The above defined  $a + 1$  mutually orthogonal maximally entangled states  $\{U_i\}$  are 2-LOCC but not 1-LOCC distinguishable.*

*Proof.* We will prove the case of  $r = 1$  without loss of generality, because the similar proof method can work for other cases of  $r > 1$ . To show the 1-LOCC indistinguishability of the  $a + 1$  maximally entangled states, we assume there exists a POVM measurement  $\{M_i\}$  with every operators  $M_k$  all rank-1 to discriminate the above set of states perfectly, which is the necessary and sufficient condition<sup>15</sup> of 1-LOCC distinguishability. That is, if suppose  $M_k$ 's diagonal sub-matrices are  $\{A_k, B_k\}$ , we should have  $Tr(U_j^\dagger M_k U_i) = \delta_i \overline{\delta_j} Tr(A_k S_i S_j^\dagger) + Tr(B_k P_i P_j^\dagger) = 0, i \neq j$  which implies

$$\delta_i \overline{\delta_j} Tr(A_k S_i S_j^\dagger) = -Tr(B_k P_i P_j^\dagger). \quad (4)$$

Because the specific forms of  $S_i, i \in \mathbb{Z}_a$  and  $P_i, i \in \mathbb{Z}_{a+1}$ , we can write Eq. (4) partly in detail.

$$\begin{cases} \delta_0 \overline{\delta_j} Tr(A_k \mathcal{X}_a^{-j}) = -Tr(B_k \mathcal{X}_{a+1}^{-j}), & j = 1, \dots, \left\lfloor \frac{a-1}{2} \right\rfloor. \\ \delta_j \overline{\delta_{2j}} Tr(A_k \mathcal{X}_a^{-j}) = -Tr(B_k \mathcal{X}_{a+1}^{-j}), & j = 1, \dots, \left\lfloor \frac{a-1}{2} \right\rfloor. \end{cases}$$

So we have  $Tr(A_k \mathcal{X}_a^{-j}) = 0$  due to the first two eqnarrays in Eq. (3), and further  $Tr(A_k \mathcal{X}_a^j) = 0$ , for  $j = 1, \dots, \left\lfloor \frac{a-1}{2} \right\rfloor, a - \left\lfloor \frac{a-1}{2} \right\rfloor, \dots, a - 1$ , by employing the hermiticity of  $A_k$ . Similarly,  $Tr(B_k \mathcal{X}_{a+1}^j) = 0$ , for  $j = 1, \dots, \left\lfloor \frac{a-1}{2} \right\rfloor, a + 1 - \left\lfloor \frac{a-1}{2} \right\rfloor, \dots, a$ . If  $a$  is odd, re-applying Eq. (4) with  $i = \frac{a+1}{2}, j = 0$ , we can obtain  $Tr(B_k \mathcal{X}_{a+1}^{(a+1)/2}) = 0$ . While if  $a$  is even, we need the third condition in Eq. (3) to obtain  $Tr(B_k \mathcal{X}_{a+1}^j) = 0$  for  $j = \frac{a}{2}, \frac{a+2}{2}$ . Therefore,  $Tr(B_k \mathcal{X}_{a+1}^j) = 0, j = 1, \dots, a$ .

Reapplying Eq. (4) with  $i = 0, \dots, a - 1; j = a$ , there will be  $\delta_i \overline{\delta_a} Tr(A_k \mathcal{X}_a^i \mathcal{Z}_a^\dagger) = -Tr(B_k \mathcal{X}_{a+1}^{i-a}) = 0$ , that is,  $Tr(S_i A_k S_j^\dagger) = 0$  for  $\forall i \neq j$  and also  $\sum_k A_k = I_a$ . It is a contradiction to the local indistinguishability of  $a + 1$  maximally entangled states in  $a \otimes a$  system. So the above set is one-way LOCC indistinguishable.

Then we will prove the 2-LOCC distinguishability through presenting a detailed protocol as follows, which is based on the protocol given in ref. 15 but more generalized and simplified.

*Step 1.* Alice does the measurement with  $\{A_k^T\}$ , where

Dimension	$p - 1 (p \geq 5)$	$np - 1 (n \geq 3, p \geq 2)$
Number of MESs	3	$p$ or $n$

**Table 1.** The relationship between the dimension and number of MESs belonging to 2-LOCC sets.

$$A_k = \frac{1}{a + 1} \sum_{j=0}^{a-1} |j\rangle \langle j + |k + a\rangle \langle k + a|.$$

If her outcome is  $k$ , then the original  $a + 1$  local unitaries  $U_i$  will become

$$V_i = \begin{pmatrix} \frac{1}{\sqrt{a + 1}} \delta_i S_i & \\ & |k \oplus_{a+1} i\rangle \langle k| \end{pmatrix}.$$

*Step 2.* To discriminate the present  $a + 1$  states with certainty, Bob should find out rank-1 measurement operators, one column of  $W$ , such that

$$(u^\dagger, v^\dagger)(V_j^T)^\dagger V_i^T \begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{5}$$

i.e.,  $\frac{1}{a+1} \overline{\delta_i} \delta_j u^\dagger \overline{S_i} S_j^T u + \overline{v_{k \oplus_{a+1} i}} v_{k \oplus_{a+1} j} = 0$ . As is referred, the eigenvectors  $|e_l\rangle$  of  $S_a$  can make  $\langle e_l | \overline{S_i} S_j^T | e_l \rangle = 0$ , for  $i < j$  and  $(i, j) \neq (k, k \oplus_{a+1} a)$ . As a result, we can choose  $v_{k \oplus_{a+1} 1} = \dots = v_{k \oplus_{a+1} a-1} = 0$  and  $u = \pm \sqrt{a + 1} |e_l\rangle$ . So according to Eq. (5), if we take  $v_k = \lambda_l$ , where  $\lambda_l$  is the eigenvalue of  $S_a$  corresponding to  $|e_l\rangle$ , then  $v_{k \oplus_{a+1} a} = -\delta_k \overline{\delta_{k \oplus_{a+1} a}}$ . The specific forms of  $\sqrt{2(a + 1)}(u^\dagger, v^\dagger)$ s are shown below

$$W = \left\{ \begin{pmatrix} \pm \sqrt{a + 1} |e_0\rangle \\ 0: \\ \lambda_0 \\ 0: \\ -\delta_k \overline{\delta_{k \oplus_{a+1} a}} \end{pmatrix}, \dots, \begin{pmatrix} \pm \sqrt{a + 1} |e_{a-1}\rangle \\ 0: \\ \lambda_{a-1} \\ 0: \\ -\delta_k \overline{\delta_{k \oplus_{a+1} a}} \end{pmatrix} \right\}.$$

What's more, we make  $u = 0$  and

$$v = \begin{cases} |a + i\rangle & i \neq k, k \oplus_{a+1} a \\ \sqrt{\frac{1}{a + 1}} |a + i\rangle & i = k, k \oplus_{a+1} a \end{cases}.$$

Therefore, the sum of all these rank-1 operators is

$$WW^\dagger = \sum \begin{pmatrix} u \\ v \end{pmatrix} (u^\dagger, v^\dagger) = \mathcal{I}_{a+(a+1)}.$$

*Step 3.* According to the necessary and sufficient condition in ref. 15, Alice can distinguish the present states based on the measurement outcome of Bob. □

Furthermore, if we choose  $S_i = \mathcal{Z}_a^i, i = 0, \dots, a - 1$  and  $S_a = \mathcal{X}_a$ , the new set of maximally entangled states also has the property of 2-LOCC but not 1-LOCC distinguishability approaching a similar proving method.

**General existence of 2-LOCC sets.** Based on the analysis of all dimension numbers previously, we only need to construct 2-LOCC sets in the quantum system of dimensions  $d \in \{np - 1, n \geq 3\}$  actually. The above “ $a + 1$ ” orthogonal MESs in the quantum system of  $[(a + 1)r + a] \otimes [(a + 1)r + a]$  can help us to explain the existence of 2-LOCC sets in  $d \otimes d$  quantum system with  $d \in \{np - 1, n \geq 3\}$ , which will prove the fact that 2-LOCC sets are ubiquitous regardless of the dimension of quantum system.

As has already pointed out,

$$np - 1 = p(n - 1) + (p - 1) = n(p - 1) + (n - 1), \quad n \geq 3,$$

we suppose either  $a = p - 1, r = n - 1$  or  $a = n - 1, r = p - 1$ , and both cases work successfully. That means, there are at least two 2-LOCC sets can be built in the quantum system of  $(np - 1) \otimes (np - 1)$ . If the dimension number plus one, i.e.,  $np$ , have two or more decompositions, then we have more choices to find out 2-LOCC sets. The relationship between the dimension number and the cardinality of a 2-LOCC set in the corresponding quantum system can be shown completely in Table 1. That is, if the dimension is  $p - 1 (p \geq 5)$ , then there exists a 2-LOCC sets including 3 MESs<sup>15</sup>. While if the dimension can be decomposed to  $np - 1 (n \geq 3, p \geq 2)$ , then there exist at least two 2-LOCC sets of  $p$  or  $n$  MESs. Until now, we can claim the fact that 2-LOCC sets are general existed regardless of the dimension of quantum system.



15. Nathanson, M. Three maximally entangled states can require two-way local operations and classical communication for local discrimination. *Phys. Rev. A* **88**, 062316 (2013).
16. Tian, G. J., Yu, S. X., Gao, F., Wen, Q. Y. & Oh, C. H. Local discrimination of four or more maximally entangled states. *Phys. Rev. A* **91**, 052314 (2015).
17. Singal, T., Rahman, R., Ghosh, S. & Kar, G. Complete Analysis of Perfect Local Distinguishability of Ensemble of Four Generalized Bell States in  $\mathbb{C}_4 \otimes \mathbb{C}_4$ , arXiv:1506.03667v1 (2015).
18. Yu, N. K., Duan, R. Y. & Ying, M. S. Any  $2 \otimes n$  subspace is locally distinguishable, *Phys. Rev. A* **84**, 012304 (2011).
19. Yu, N. K., Duan, R. Y. & Ying, M. S. Distinguishability of quantum states by positive operator-valued measures with positive partial transpose. *IEEE Trans. Inf. Theory* **60**, 2069 (2014).
20. Yu, N. K., Duan, R. Y. & Ying, M. S. Four Locally Indistinguishable Ququad-Ququad Orthogonal Maximally Entangled States. *Phys. Rev. Lett.* **109**, 020506 (2012).
21. Cosentino, A. Positive-partial-transpose-indistinguishable states via semidefinite programming. *Phys. Rev. A* **87**, 012321 (2013).
22. Cosentino, A. & Russo, V. Small sets of locally indistinguishable orthogonal maximally entangled states. *Quantum Inf. Comput.* **14**, 1098–1106 (2014).

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## Author Contributions

G.T., X.W., Y.C. and F.G. initiated the idea. G.T., F.G. and Q.W. wrote the main manuscript. All authors reviewed the manuscript.

## Additional Information

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