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Skyrmion Superfluidity in Two-Dimensional Interacting Fermionic Systems

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In this article we describe a multi-layered honeycomb lattice model of interacting fermions which supports a new kind of parity-preserving skyrmion superfluidity. We derive the low-energy field theory describing a non-BCS fermionic superfluid phase by means of functional fermionization. Such effective theory is a new kind of non-linear sigma model, which we call double skyrmion model. In the bi-layer case, the quasiparticles of the system (skyrmions) have bosonic statistics and replace the Cooper-pairs role. Moreover, we show that the model is also equivalent to a Maxwell-BF theory, which naturally establishes an effective Meissner effect without requiring a breaking of the gauge symmetry. Finally, we map effective superfluidity effects to identities among fermionic observables for the lattice model. This provides a signature of our theoretical skyrmion superfluidity that can be detected in a possible implementation of the lattice model in a real quantum system.

Quantum field theory (QFT) plays a fundamental role in the description of strongly correlated systems and topological phases of matter. For example, free and self-interacting relativistic fermions emerging in condensed matter systems can be described by Dirac and Thirring theories respectively^{1–5}. At the same time, the ground states of fractional quantum Hall states, topological insulators and superconductors are opportunely described by bosonic topological QFTs like Chern-Simons and BF theories^{6–9}. Another class of bosonic QFT contains the non-linear sigma models (NLSM) which describe the physics of Heisenberg antiferromagnets¹⁰, Quantum Hall ferromagnets¹¹ and symmetry protected topological phases^{12,13}. The addition of a topological term in the theory (Hopf term)¹⁴ allows for the skyrmions (the quasiparticles present in the model) to acquire fermionic, bosonic or anyonic statistics depending on the value of the coefficient in front of the Hopf term and the value of their topological charge^{15,16}. Importantly, bosonic QFTs reveal several features which characterize the physics of superconductivity. In particular, skyrmions appear as topological defects in three-band superconductors¹⁷, in Bose-Einstein condensations¹⁸ and have been used to define and describe a parity-breaking two-dimensional non-BCS superconductivity^{19–21}, while BF theory is a candidate as the effective theory for some strongly correlated fermionic systems⁵, graphene²² and spin Hall states^{20,23}. BF theory naturally describe the Meissner effect^{9,24,25}, which represents the smoking-gun evidence of superconducting phase. These non-BCS superconducting mechanisms could be used to get insights on the physics of high-temperature superconductors^{24,26}.

The goal of this letter is to provide a new fermionic (multi-layered) honeycomb lattice model that combines characteristics of *both* skyrmions and BF theory in an unified way. This allows us to prove the existence of a parity-preserving non-BCS superfluid phase (analog neutral version of superconducting phase). More specifically, as a consequence of a detailed field theory derivation, we prove that our model supports the emergence of *both* an effective Meissner effect *and* the formation of Cooper-like pairs. This is the ground on which we build the other main result of this work. In fact, as the proposed tight-binding model is plausible enough to allow for future experimental investigations, we rigorously prove a map between physical fermionic observables and effective bosonic ones. We show that these observables have

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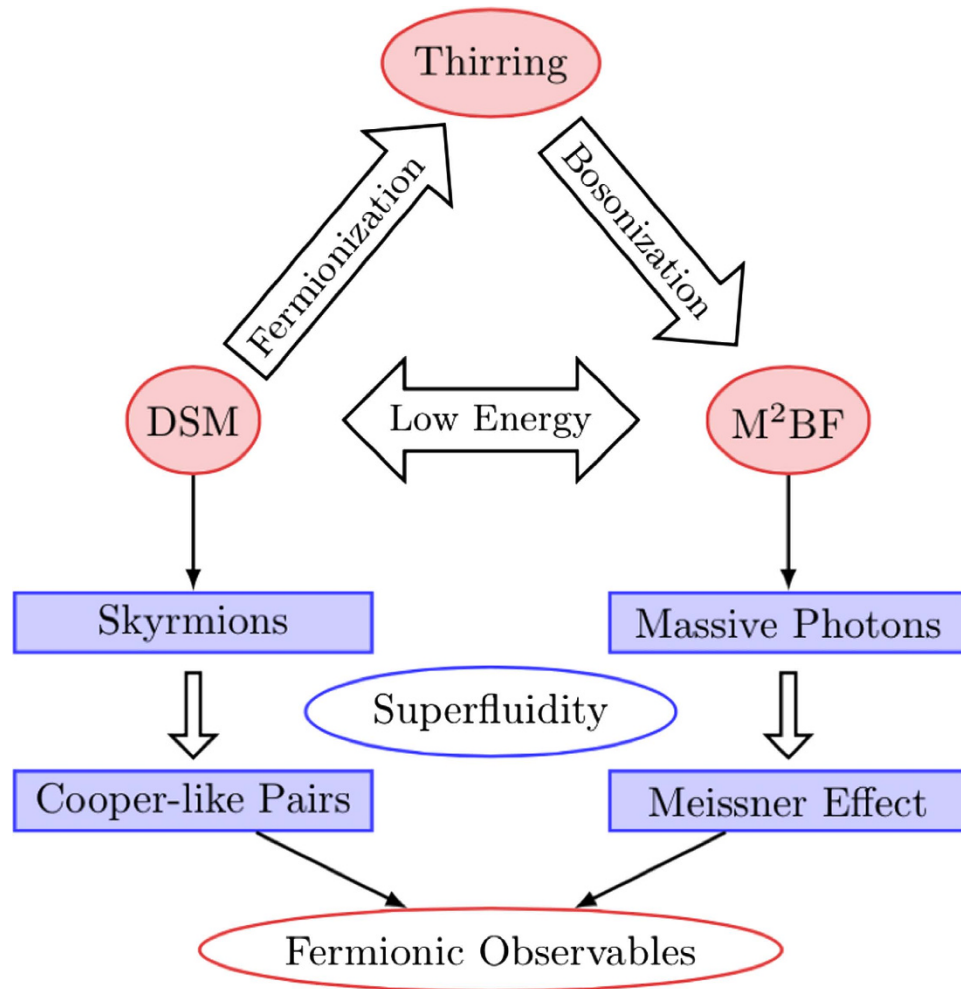


Figure 1. Sketch of the logical structure among the effective field theories describing the model and the corresponding physical properties associated to fermionic superfluidity.

to satisfy explicit relations, consistently with *both* the emergent properties of the model. In this way our model prepares the way for an experimental probe of its emergent superconducting properties.

The logical structure of the article is sketched in Fig. 1. Specifically, the system is described by a fermionic Hubbard-like model which gives rise, in the low-energy limit, to a $(2+1)$ -dimensional chiral-invariant Thirring model²⁷ supporting self-interacting Dirac particles. By using functional fermionization techniques^{26,28}, we show that this theory is equivalent to a new kind of skyrmion model which is invariant under parity and time-reversal transformations. We call it double skyrmion model (DSM). Interestingly, the statistics of the skyrmions can depend on the number of layers. For bi-layer systems skyrmions behave as (neutral) bosons and represent the natural Cooper-like pairs in the (fermionic) superfluid phase. In addition, we show that the system can also be described by a double(Maxwell)-BF (M^2BF) theory which is a particular instance of a topologically massive gauge theory (TMGT). This equivalence can be shown either by integration of the scalar skyrmionic field or directly from the fermionic Thirring model by means of functional bosonization²⁹. In the TMGT theory, effective photons acquire a mass as a consequence of topological interactions. This naturally leads to the London equations of superconductivity (fermion superfluidity)²⁴ which effectively combine Meissner effect and infinite conductivity. We finally show how physical fermionic observables can probe the skyrmion superfluid mechanism described by the model.

Lattice Model

We consider n two-dimensional layers of spinful fermions stacked on the top of each other (Fig. 2). Within each layer fermions are localized on a honeycomb lattice. In the case $n = 1$ the fermion hopping is described by the following graphene-like (spin $s = \uparrow, \downarrow$ dependent) Hamiltonian.

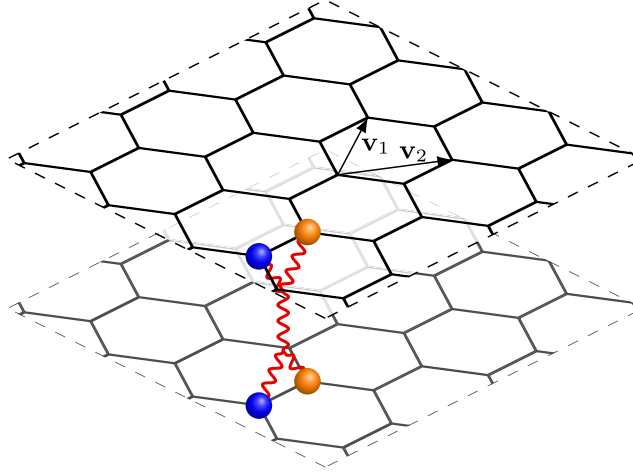


Figure 2. Tight binding for $n=2$. Fermions hop along the edges of two honeycomb lattice layers as described by the Hamiltonian in Eq. (1). For each spin species and each layer the unit cell contains two fermions: a (orange) and b (blue). Fermions in different layers but in unit cells with same in-layer coordinates interact through current-current interactions (wavy line) as described by the Hamiltonian in Eq. (3).

$$H_0 = \pm c \sum_{s,r} \left[(a_r^{s\dagger} b_{r+v_1}^s + a_r^{s\dagger} b_{r+v_2}^s + a_r^{s\dagger} b_r^s) + mc (a_r^{s\dagger} a_r^s - b_r^{s\dagger} b_r^s) \right]. \tag{1}$$

Here, the overall sign depends on the orientation of the spin and a_r and b_r are the fermion operators at position $\mathbf{r} \in \Lambda$ where $\Lambda = n_1 \mathbf{v}_1 + n_2 \mathbf{v}_2$ is the lattice of unit cells of the model ($n_1, n_2 \in \mathbb{N}$ and $\mathbf{v}_1 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$ and $\mathbf{v}_2 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$). This Hamiltonian describes hopping terms along the links of the honeycomb lattice with a real tunneling coefficient c and a staggered chemical potential (with energy scale mc^2) and it can be exactly solved. The spectrum becomes gapless at two independent points P_{\pm} in momentum space. The low energy physics around these points is effectively described by a standard massive Dirac Hamiltonian

$$H_0 = \sum_{\pm} \int d^2k \left[\Psi_{\pm}^{\dagger} (ck_x \alpha^x + ck_y \alpha^y + mc^2 \beta) \Psi_{\pm} \right], \tag{2}$$

where the matrices α and β belong to an euclidean Clifford algebra and where, for clarity, the energy scales c and mc^2 have been renormalized (for details, see appendix). The spinors Ψ_{\pm} depend on the momentum space coordinate k as $\Psi_{\pm} = (a_{\pm}^{\dagger}(k^{\pm}) \ b_{\pm}^{\dagger}(k^{\pm}) \ a_{\pm}^{\downarrow}(k^{\pm}) \ b_{\pm}^{\downarrow}(k^{\pm}))^T$ where a_{\pm} are the Fourier transformed fermion operators evaluated at the Fermi points P_{\pm} respectively and where $k^+ = (k_x, k_y)$ and $k^- = (-k_x, k_y)$. Note that it is possible to induce the same mass term in the above Hamiltonian by replacing the staggered chemical potential in (1) with a standard Haldane term³⁰.

We now consider the general case of n such layers (we will be mainly interested in the case $n=2$) and label their free Hamiltonians by $j=1, \dots, n$ so that $H_0 \rightarrow H_0^j$. To connect the layers we add current-current interactions to the free model

$$H = \sum_{j=1}^n H_0^j + H_I, \tag{3}$$

with $H_I = \left[\frac{g^2}{2} (\sum_j J_{\mu}^j)^2 + (\sum_j J_{\mu}^{j5})^2 \right]$, where $\mu = 0, 1, 2$, where the spinor and the currents are, respectively, $\Psi_j = (a^{j\dagger} \ b^{j\dagger} \ a^{j\downarrow} \ b^{j\downarrow})^T$, $J^{\mu} = \bar{\Psi}_j \gamma^{\mu} \Psi_j$ and $J^{5\mu} = \bar{\Psi}_j \gamma^5 \gamma^{\mu} \Psi_j$, and where the γ s are Dirac gamma matrices. In the case of a single layer, we have that

$$H_I = 3g^2 [|(a^{\dagger} b^{\dagger} + a^{\downarrow} b^{\downarrow})|^2 + |(a^{\dagger} b^{\downarrow} - a^{\downarrow} b^{\dagger})|^2]. \tag{4}$$

The less compact, but similar, expression for the case $n=2$ can be found in the Supplemental Material. Around each Fermi point P_{\pm} the low-energy effective physics is described by the following partition function $ZF = \int \mathcal{D}\bar{\Psi}_j \mathcal{D}\Psi_j e^{\pm SF}$, where $SF = S_0 + S_I$ with

$$S_0 = \sum_{j=1}^n \int d^3x \bar{\Psi}_j (ci\not{\partial} - mc^2) \Psi_j \quad S_I = \int d^3x \frac{g^2}{2} \left[\left(\sum_{j=1}^n J_\mu^j \right)^2 + \left(\sum_{j=1}^n J_\mu^{j5} \right)^2 \right] \quad (5)$$

This model is nothing but a (generalized) chiral-invariant Thirring model²⁷. In the following we will work in units such that $c = \hbar = 1$ and without losing of generality we will consider the physics only around one Fermi point.

Double skyrmion model and functional fermionization. We now introduce a double skyrmion model (DSM) which is a double O(3)-Hopf non-linear sigma model

$$S_{(O(3)-H)^2} = \sum_{t=\pm} \int d^3x \left[\frac{1}{2g_0^2} (\partial_\mu \mathbf{m}^t \partial^\mu \mathbf{m}^t) + tn\pi H^t \right], \quad (6)$$

where n coincides with the number of layers, the fields $\mathbf{m}^\pm \in \mathbb{R}^3$ satisfy the non-linear constraint $|\mathbf{m}^\pm|^2 = 1$ and the two Hopf terms H^t are topological invariants¹⁴. Due to the different sign in front of the Hopf terms, this theory describes independent skyrmions and anti-skyrmions which have opposite values of the topological charges $Q_T^+ = -Q_T^-$ which assume only integer values (see Supplemental Material). Each (anti-)skyrmion has a spin S given by³¹

$$S = (n/2) Q_T^{\pm 2}. \quad (7)$$

This shows that, depending on the number of layers and value of topological charge, the statistics of the skyrmions can be either bosonic or fermionic. In particular, for a bi-layer system, (anti-)skyrmions behave like bosons for any value of Q_T^\pm and in our context take the role Cooper-like pairs.

Following²⁶, we now use functional *fermionization* to show the equivalence between the partition function of this bosonic theory and the one describing a chiral-invariant Thirring model. Let us start by defining the equivalent CP form⁷ of the O(3) NLSM in Eq. (6) in which the Hopf terms are recast as Chern-Simons terms. We will refer to it as a double CP-Chern-Simons (CP-CS)² model

$$Z_{(CP-CS)^2} = \int \mathcal{D}A^+ \mathcal{D}A^- \mathcal{D}z^+ \mathcal{D}z^- e^{iS_{(CP-CS)^2}}, \quad (8)$$

where

$$S_{(CP-CS)^2} = \sum_{t=\pm} \int d^3x \left[\pm \frac{n}{4\pi} \varepsilon^{\lambda\mu\nu} A_\lambda^t \partial_\mu A_\nu^t + \frac{1}{g^2} \left| (i\partial_\mu - A_\mu^t) z^t \right|^2 \right]. \quad (9)$$

Here, $z^\pm = (z_1^\pm, z_2^\pm)^T$ with the fields $z_1^\pm, z_2^\pm \in \mathbb{C}$ such that $|z^\pm|^2 = z_1^\pm z_1^{\pm*} + z_2^\pm z_2^{\pm*} = 1$ and $A_\mu^\pm = - (i/2) z^{\pm*} \partial_\mu z^\pm$. We now proceed with the fermionization (see Supplemental Material). The fermions appear quite naturally. In fact, we begin by noticing that, by changing variables to $A_\mu = \frac{1}{2}(A_\mu^+ + A_\mu^-)$, $B_\mu = \frac{1}{2}(A_\mu^+ - A_\mu^-)$, the difference of the two Chern-Simons terms can be written as a BF term $\frac{n}{\pi} \int d^3x \varepsilon^{\mu\nu\lambda} B_\mu \partial_\nu A_\lambda$. This leads to a (double CP)-BF theory so that $Z_{CP^2-BF} = Z_{(CP-CS)^2}$. The BF term can now be “linearized” by introducing²⁴ n fermion species χ_j leading to the following intermediate partition function

$$Z = \int \mathcal{D}A \mathcal{D}B \mathcal{D}z^+ \mathcal{D}z^- \mathcal{D}\chi \mathcal{D}\bar{\chi} \exp \left\{ i \sum_j \int d^3x \left[\bar{\chi}_j (i\not{\partial} - m) \chi_j - \sqrt{2} A_\mu J_\mu^j - \sqrt{2} B_\mu J_\mu^{5j} \right] + \frac{1}{g^2} \sum_{t=\pm} \left| (i\partial_\mu - (A_\mu + tB_\mu)) z^t \right|^2 \right\}. \quad (10)$$

For each value of the sign, the variable z^\pm can be thought as specifying a coordinate system in a SU(2) algebra via the identification $z^\pm \rightarrow e^{i\xi_j^\pm \sigma^j}$, where σ^j are the Pauli matrices and ξ_j^\pm are scalar fields, see Supplemental Material. Moreover, the gaussian integral over the fields A and B can be easily computed. This cause the fields ξ_j^\pm to effectively decouple. A change of fermionic variables $\chi_j \rightarrow \Theta \Psi_j$ with a suitable phase Θ (see Supplemental Material) leads directly to

$$ZF = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp \left\{ i \sum_j \int d^3x \left[\bar{\Psi}_j (i\partial - m) \Psi_j + \frac{g^2}{2} (J_\mu^j)^2 + \frac{g^2}{2} (J_\mu^{5j})^2 \right] \right\}, \quad (11)$$

which is in fact the original chiral-invariant Thirring model introduced in the previous section. As a final comment, we note that, alternatively, it is possible to *bosonize* the fermionic model in Eq. (11) to the (CP-CS)² model, see Supplemental Material.

London action. In this section we show that the effective theory described in Eq. (8) is equivalent to the London action, which effectively describes the physics of superconductivity.

In^{24,32} it is proven that (at low energy) a CP model is equivalent to a Maxwell theory. We can use this to map the effective theory in Eq. (8) to a (double Maxwell)-BF theory (M²BF)^{9,24} with action

$$S_{M^2\text{-BF}} = \int d^3x \left[\frac{n}{\pi} \varepsilon^{\lambda\mu\nu} B_\lambda \partial_\mu A_\nu - \frac{1}{4e^2} F_{\mu\nu}(A) F^{\mu\nu}(A) - \frac{1}{4e^2} F_{\mu\nu}(B) F^{\mu\nu}(B) \right], \quad (12)$$

where $F_{\mu\nu}$ is the field strength tensor and the two scales g_0 and e can be explicitly related, see Supplemental Material. We now follow Ref. 24 (Supplemental Material) which show that this theory is equivalent to the London partition function:

$$Z_\phi = \int \mathcal{D}A \mathcal{D}\phi e^{i \int d^3x \left[-\frac{1}{4e^2} F_{\mu\nu}(A) F^{\mu\nu}(A) + 2e^2 \left(\partial_\mu \phi - \frac{n}{2\pi} A_\mu \right)^2 \right]}. \quad (13)$$

We can see that the (2+2) degrees of freedom of the massless fields A and B are mapped to the (3+1) degrees of freedom of a massive bosonic field A and a massless scalar field is ϕ which, in this sense, represents a kind of Goldstone boson. The present mechanism, however, does not have any local order parameter like in ordinary BCS theory. The charge and currents associated with the field A are

$$\rho = \frac{\delta \mathcal{L}_\phi}{\delta A_0} \quad \mathbf{J}_{\text{em}} = \frac{\delta \mathcal{L}_\phi}{\delta \mathbf{A}}, \quad (14)$$

where \mathcal{L}_ϕ is the Lagrangian density associated with Z_ϕ . The effective magnetic and electric fields inside the material are simply given by

$$E^i = \frac{1}{2} \varepsilon^{i\mu\nu} F(A)_{\mu\nu} \quad B_{\text{mag}} = \frac{1}{2} \varepsilon^{0\mu\nu} F(A)_{\mu\nu}, \quad (15)$$

where $i = x, y$. The effective physics described by the massive field A , implies both a Meissner and infinite conductivity effects. In fact, the (effective) magnetic field intensity decays exponentially inside the material (Meissner effect) due to the presence of superficial dissipationless screening currents. In particular we have that

$$B_{\text{mag}} = 0 \quad (16)$$

in the bulk of the material. As shown in the literature^{24,33} a zero voltage can be defined in the presence of steady currents. These screening currents flow within a penetration depth $\lambda \propto sg^2$ (see Supplemental Material) from the boundary of the material. In this sense, the system has infinite conductivity σ and follows the perfect conductivity relation $\mathbf{E} = \sigma \mathbf{J}_{\text{em}}$.

Fermionization rules and physical observables. The aim of this section is to map the effective superfluidity physics that describes the model to fermionic observables. To this end, we introduce a minimal coupling interaction with two external fields A_{ext} and B_{ext} to the fermionic Lagrangian density inside Eq. (11) via a minimal coupling $\sum_j j_\mu^j A_{\text{ext}}^\mu + \sum_j j_\mu^{5j} B_{\text{ext}}^\mu$ so that $ZF \rightarrow ZF(A_{\text{ext}}, B_{\text{ext}})$. We then track these new terms as we follow back all the steps that lead us from Eq. (10) to Eq. (11). The additional terms only cause a shift in the Dirac operator $ZF(A_{\text{ext}}, B_{\text{ext}}) = Z_{\text{CP}^2\text{BF}}(A_{\text{ext}}, B_{\text{ext}})$ which leads to the equivalence $ZF(A_{\text{ext}}, B_{\text{ext}}) = Z_{\text{CP}^2\text{BF}}(A_{\text{ext}}, B_{\text{ext}})$, where this last partition function has an additional term $\frac{n}{\sqrt{2}\pi} \int \varepsilon^{\mu\nu\lambda} \left(-B_\mu^{\text{ext}} \partial_\nu A_\lambda + A_\mu^{\text{ext}} \partial_\nu B_\lambda + \frac{1}{\sqrt{2}} B_\mu^{\text{ext}} \partial_\nu A_\lambda^{\text{ext}} \right)$ in the action, (see Supplemental Material). By taking derivatives of the partition functions with respect to the external fields (see Supplemental Material) this allows us to prove the following “fermionization” rules.

$$\begin{aligned} \langle \sum_j J_\mu^j \rangle F &= \frac{n}{\sqrt{2}\pi} \langle \varepsilon_{\mu\nu\lambda} \partial_\nu B_\lambda \rangle_{\text{CP}^2\text{BF}} \\ \langle \sum_j J_\mu^{5j} \rangle F &= -\frac{n}{\sqrt{2}\pi} \langle \varepsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \rangle_{\text{CP}^2\text{BF}}, \end{aligned} \quad (17)$$

where the expectation values $\langle \rangle F$ and $\langle \rangle_{\text{CP}^2\text{BF}}$ are calculated with respect to the ground state of the fermionic and bosonic theory respectively.

Similarly, we can map observables for the London theory to observables for the double(Maxwell)-BF by adding and tracking source terms $F_{\mu\nu}(A)\varepsilon^{\mu\nu\lambda}J_\lambda^A$ and $F_{\mu\nu}(B)\varepsilon^{\mu\nu\lambda}J_\lambda^B$ to the latter theory (see Supplemental Material) so that $Z_{M^2BF} \rightarrow Z_{M^2BF}(J^A, J^B)$.

We can use these correspondences to relate the current (ρ, \mathbf{J}_{em}) and the fields (B_{mag}, \mathbf{E}) defined in Eqs. (14) and (15) to fermionic observables. Inspection of Eq. (14) and the expression of $Z_\varphi(J^A, J^B)$ immediately leads (see Supplemental Material) to $\langle \rho \rangle_\phi = \frac{2\pi}{n} \frac{\delta Z_\phi}{\delta j_0^B} = \frac{2\pi}{n} \frac{\delta Z_{M^2BF}}{\delta j_0^B}$ and $\langle \mathbf{J}_{em} \rangle_\phi = \frac{2\pi}{n} \frac{\delta Z_\phi}{\delta j_i^A} = \frac{2\pi}{n} \frac{\delta Z_{M^2BF}}{\delta j_i^B}$ (where we implicitly imposed $J^{A,B} = 0$, after the derivative is taken).

Finally, using the fermionization rules in Eq. (17) and the equivalences among London, (double Maxwell)-BF and (double CP)-BF theories we find (see Supplemental Material) the promised relation

$$\begin{aligned} \langle \rho \rangle_\phi &= \langle \varepsilon^{0\mu\nu} F_{\mu\nu}(B) \rangle_{M^2BF} = \frac{2\sqrt{2}\pi}{n} \langle \sum_j J_0^j \rangle_F \\ \langle \mathbf{J}_{em}^i \rangle_\phi &= \langle \varepsilon^{i\mu\nu} F_{\mu\nu}(B) \rangle_{M^2BF} = \frac{2\sqrt{2}\pi}{n} \langle \sum_j J_i^j \rangle_F. \end{aligned} \tag{18}$$

A parallel procedure can be applied to the electric and magnetic fields to get $\langle E^i \rangle_\phi = \frac{1}{2} \frac{\delta Z_\phi}{\delta j_i^A} = \frac{\sqrt{2}\pi}{n} \langle \sum_j J_i^{5j} \rangle_F$ and $\langle B_{mag} \rangle_\phi = \frac{1}{2} \frac{\delta Z_\phi}{\delta j_0^A} = \frac{\sqrt{2}\pi}{n} \langle \sum_j J_0^{5j} \rangle_F$. These findings are summarized in the following table.

Electromagnetic Quantities	Fermionic Observables
$\langle \rho \rangle_\phi$	$\langle \sum_j J_0^j \rangle_F$
$\langle \mathbf{J}_{em} \rangle_\phi$	$\langle \sum_j \mathbf{J}^j \rangle_F$
$\langle \mathbf{E} \rangle_\phi$	$\langle \sum_j \mathbf{J}^{5j} \rangle_F$
$\langle B_{mag} \rangle_\phi$	$\langle \sum_j J_0^{5j} \rangle_F$

This table allows us to write the effective Meissner effect in Eq. (16) in terms of fermionic observables as

$$\langle \sum_j J_0^{5j} \rangle_F = 0. \tag{19}$$

The validity of such a prediction is confirmed by the skyrmionic interpretation of the model. In fact, Eq. (19) can be derived from an alternative dual point of view. We first notice that the expectation value of the skyrmion currents (see Supplemental Material) $J_S^{\mu\pm} = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda^\pm$ can be written as $\langle J_S^\pm \rangle_{(CP-CS)^2} = \pm \frac{1}{n\sqrt{2}} \left(\langle \sum_j J^j \rangle_F \mp \langle \sum_j J^{5j} \rangle_F \right)$ by using the fermionization rules in Eq. (17). Now, by definition, the topological charges are the spatial integral of the 0th component of the current $Q_T^\pm = \int d^2x J_S^{0\pm}$ so that

$$0 = - \frac{n}{\sqrt{2}} (Q_T^+ + Q_T^-) = - \frac{n}{\sqrt{2}} \int d^2x \sum_{t=\pm} \langle J_S^{0t} \rangle_{(CP-CS)^2} = \int d^2x \langle \sum_j J_0^{5j} \rangle_F, \tag{20}$$

consistently with Eq. (19).

At the same time, as mentioned above, the system supports steady state currents within a penetration depth $\lambda \propto g^2$ distance from the boundary. By tuning the parameter g to allow the fermionization rules to hold, the Drude relation $\mathbf{E} = \sigma \mathbf{J}_{em}$ maps (see Supplemental Material) to the fermionic constraint $\langle \sum_j \mathbf{J}^{5j} \rangle = \sigma \langle \sum_j \mathbf{J}^j \rangle$ where $\sigma \rightarrow \infty$.

Conclusions

In this article we proposed a fermionic tight-binding model which naturally supports the two main ingredients of fermionic superconductivity: Cooper-like pair formation and Meissner effect. In order to prove these effects, we employed functional fermionization to show the equivalence between the effective fermionic theory describing the lattice system (a chiral-invariant Thirring model) and a double skyrmion model. This model supports skyrmions with bosonic statistics (Cooper-like pairs) in the bi-layer case and it is formally equivalent to a double Maxwell-BF theory which describes an effective Meissner effect. Moreover, we rigorously mapped (fermionic) physical observables to effective (bosonic) ones. In this way, we found explicit identities among the physical observables which appear as a direct consequence of both the presence of Cooper-like pairs and the Meissner effect. These relations are crucial to detect a signature of the effective physics in a possible implementation of the lattice model in a real (or simulated³⁴⁻³⁶) quantum system. This could lead to the interesting possibility to experimentally probe superfluidity properties in an highly controlled physical setting (like cold atoms) opening the road to new possible applications and explorations of this physics.

A straightforward generalization of our model to the (charged) superconducting case can be obtained once neutral fermions are replaced with charged ones and an external electromagnetic field coupled with

them is taken into account. Finally, an open question related to this work concerns the possible existence of Abrikosov-like vortices and the presence of Majorana states localized at their cores^{37–39}. We leave the study of these important aspects to future works.

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