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Effects of reciprocity on random walks in weighted networks

Zhongzhi Zhang^{1,2}, Huan Li^{1,2} & Yibin Sheng^{1,2}¹School of Computer Science, Fudan University, Shanghai 200433, China, ²Shanghai Key Lab of Intelligent Information Processing, Fudan University, Shanghai 200433, China.

It has been recently reported that the reciprocity of real-life weighted networks is very pronounced, however its impact on dynamical processes is poorly understood. In this paper, we study random walks in a scale-free directed weighted network with a trap at the central hub node, where the weight of each directed edge is dominated by a parameter controlling the extent of network reciprocity. We derive two expressions for the mean first passage time (MFPT) to the trap, by using two different techniques, the results of which agree well with each other. We also analytically determine all the eigenvalues as well as their multiplicities for the fundamental matrix of the dynamical process, and show that the largest eigenvalue has an identical dominant scaling as that of the MFPT. We find that the weight parameter has a substantial effect on the MFPT, which behaves as a power-law function of the system size with the power exponent dependent on the parameter, signaling the crucial role of reciprocity in random walks occurring in weighted networks.

As an emerging science, complex networks have witnessed substantial progress in the past years¹. One of the ultimate goals in the study of complex networks is to uncover the influences of various structural properties on the function or dynamical processes taking place on them. Among different dynamical processes, random walks lie at the core, since they are a fundamental mechanism for a wealth of other dynamic processes, such as navigation², search^{3,4}, and cooperative control⁵. Except for the importance in the area of network science, random walks also provide a paradigmatic model for analyzing and understanding a large variety of real-world phenomena, for example, animal⁶ and human⁷ mobility. Thus far, random walks have found numerous applications⁸ in many aspects of interdisciplinary sciences, including image segmentation⁹, community detection^{10,11}, collaborative recommendation¹², and signal propagation in proteins¹³ to name a few.

A highly desirable quantity for random walks is first passage time (FPT)¹⁴, defined as the expected time for a random walker going from a starting node to a given target averaged over all possible trajectories. The mean of FPTs over all starting nodes to the target is called mean first passage time (MFPT), which is an important characteristic of random walks due to the first encounter properties in numerous realistic situations. In the past years, the study of MFPT has triggered an increasing attention from the scientific community^{15,16}. One focus of theoretical activity is to develop general methods to efficiently compute MFPT^{17–20}. Another direction is to unveil how the behavior of MFPT is affected by different structural properties of the underlying systems, such as heterogeneity of degree²¹ or strength²², fractality²³, and modularity²⁴.

Previous studies proposed several frameworks for evaluating MFPT and uncovered the discernible effects of some nontrivial structural aspects on the target search efficiency measured by MFPT. However, most existing works ignore the impact of link reciprocity, the tendency of node pairs to form mutual connection in directed networks, on the behavior of random walks, despite the fact that reciprocity is a common characteristic of many realistic networks²⁵, such as the World-Wide Web²⁶, e-mail networks^{27,28}, and world trade web²⁹. In addition to binary networks, the nontrivial pattern reciprocity is also ubiquitous in real-life systems described by weighted networks^{30–32}. It has been shown the ubiquitous link reciprocity strongly affects dynamical processes in binary networks, for example, spread of computer viruses²⁸ or information³³, and percolation³⁴. By contrast, the influence of reciprocity on dynamical processes in weighted networks has attracted much less attention, although it is suggested that reciprocity could play a crucial role in network dynamics. In particular, the lack of analytical results in this field limits our understanding of the impact of weight reciprocity on the function of weighted networks³².

In this paper, we propose a weighted directed scale-free network through replacing each edge in the previous binary network^{35,36} by double links with opposite directions and different weights. In the weighted network, the link weights are adjusted by a parameter characterizing the weight reciprocity of network. We then study random walks in the weighted network in the presence of a perfect trap at the central large-degree node. During the process of random walks, the transition probability is dependent on the weight parameter. We derive two

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formulas for the MFPT to the target by using two disparate approaches, the results of which completely agree with each other. We also determine all the eigenvalues and their multiplicities of the fundamental matrix characterizing the random-walk process, and show that the largest eigenvalue has the same leading scaling as that of the MFPT. The obtained results demonstrate that the behavior of MFPT to the trap depends on the weighted parameter, implying a drastic influence of the weight reciprocity on random walks defining on weighted networks.

Results

Network models and properties. Before introducing the weighted directed network with scale-free fractal properties. We first give a brief introduction to a binary scale-free fractal network, which has the same topology as the weighted network.

Model and properties of binary network. The binary network is constructed in an iterative way^{35,36}. Let F_g ($g \geq 0$) represent the network after g iterations (generations). For $g = 0$, F_0 is an edge linked by two nodes. In each successive iteration $g \geq 1$, F_g is constructed from F_{g-1} by performing the following operations on every existing edge in F_{g-1} as shown in Fig. 1: two new nodes (called external nodes) are firstly created and attached, respectively, to both endpoints of the edge; then, the edge is broken, another new node (referred to as an internal node) is placed in its middle and linked to both endpoints of the original edge. Figure 2 illustrates the first several construction processes of the network. The structure of F_g is enciphered in its adjacency matrix A_g , the entries $A_g(i, j)$ of which are defined by $A_g(i, j) = 1$ if two nodes i and j are adjacent in F_g , or $A_g(i, j) = 0$ otherwise.

The particular construction of the network allows to calculate exactly its relevant properties. At each generation g_i ($g_i \geq 1$), the number of newly created nodes is $\Upsilon_{g_i} = 3 \cdot 4^{g_i-1}$. Let $\bar{\Lambda}_{g_i}$ be the set of nodes generated at iteration g_i , then $\bar{\Lambda}_{g_i}$ can be further classified into two sets $\bar{\Lambda}_{g_i,ext}$ and $\bar{\Lambda}_{g_i,int}$ satisfying $\bar{\Lambda}_{g_i} = \bar{\Lambda}_{g_i,ext} \cup \bar{\Lambda}_{g_i,int}$, among which $\bar{\Lambda}_{g_i,ext}$ is the set of external nodes and $\bar{\Lambda}_{g_i,int}$ is the set of internal nodes. We use $|\Omega|$ to stand for the cardinality of a set Ω . Because $|\bar{\Lambda}_{g_i,ext}| = 2|\bar{\Lambda}_{g_i,int}|$, it is easy to derive $|\bar{\Lambda}_{g_i,int}| = 4^{g_i-1}$ and $|\bar{\Lambda}_{g_i,ext}| = 2 \cdot 4^{g_i-1}$. We represent the set of nodes in F_g as Λ_g . Hence, the number of nodes and edges in F_g is $N_g = |\Lambda_g| = \sum_{g_i=0}^g \Upsilon_{g_i} = 4^g + 1$ and $E_g = N_g - 1 = 4^g$, respectively. Let $k_i(g)$ denote the degree of an arbitrary node i in F_g that was generated at generation g_i ($g_i \geq 0$), then $k_i(g+1) = 2k_i(g)$. Hence, after each new iteration the degree of every node doubles.

This resultant network displays the remarkable scale-free³⁷ and fractal³⁸ features as observed in diverse real-life systems. It has a power law degree distribution with an exponent 3, and its fractal dimension is 2.

Model and properties of weighted directed network. The above introduced binary network F_g can be extended to a weighted directed network with nonnegative and asymmetrical edge weights. Let \vec{F}_g denote the weighted directed network corresponding to F_g . Both \vec{F}_g and F_g have an identical topological structure. The only difference between \vec{F}_g and F_g is that every undirected edge in F_g is replaced by two directed edges with opposite directions and distinct positive weights. We use W_g to represent the nonnegative and asymmetrical

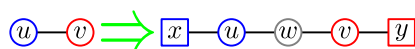


Figure 1 | Illustration of construction of the binary network. The next generation is obtained from current generation by replacing each edge with the cluster on the right-hand side of the arrow, where w is a new internal node, while x and y are external nodes.

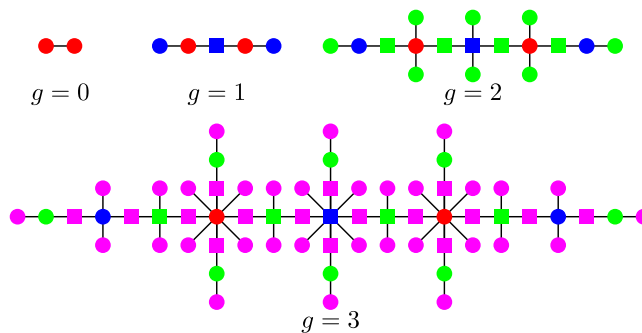


Figure 2 | Iterative growth processes for the first several generations.

weight matrix for \vec{F}_g such that $W_{ij}(g) > 0$ if and only if there is a directed edge (arc) pointing to node j from node i . The weight of each arc in the weighted directed network is defined recursively in the following way. When $g = 0$, \vec{F}_0 has two nodes, denoted by a and b , and the weights of arcs $\vec{e}(a,b)$ and $\vec{e}(b,a)$ are defined to be $W_{ab}(0) = W_{ba}(0) = 1$. When $g \geq 1$, by construction, F_g is obtained from F_{g-1} by substituting each undirected edge $e(u, v)$ in F_{g-1} with two undirected edges $e(u, w)$ and $e(w, v)$, and generating two additional nodes, x and y , attaching to u and v , respectively. The weights of resultant arcs in \vec{F}_g are defined as: $W_{uw}(g) = W_{uv}(g-1)$, $W_{vw}(g) = W_{vu}(g-1)$, $W_{wu}(g) = W_{wv}(g) = 1$, $W_{xu}(g) = 1$, $W_{yv}(g) = 1$, $W_{ux}(g) = \theta W_{uv}(g-1)$, and $W_{vy}(g) = \theta W_{vu}(g-1)$. Here θ is a tunable positive real number, that is, $\theta > 0$. The weight parameter is of paramount importance since it characterizes the weight reciprocity of network \vec{F}_g . When $\theta = 1$, \vec{F}_g reduces to F_g , and the weights in two directions between any pair of adjacent nodes are completely reciprocated; when $\theta \neq 1$, the weights are non-reciprocated³⁰⁻³²: the larger the deviation of θ from 1, the smaller the level of weight reciprocity.

In fact, the introduced weight parameter θ acts as a similar role of energetic funnel in dendrimers controlling the MFPT to the center³⁹⁻⁴¹. It has been shown theoretically^{39,42} that the MFPT of Cayley trees as models of dendrimers can be influenced by both geometrical and energetic features. However, for complex systems with scale-free and fractal properties, it is still not well-understood how to superimpose an energetic funnel on their topological architecture, so that the MFPT depends on the interplay between structural and energetic properties. The main purpose of this work is to fill the gap, uncovering the collective impacts of topological and energetic aspects on dynamical processes, especially MFPT of random walks.

In undirected weighted networks⁴³, node strength is a key quantity characterizing the property of a node. Here we extend the definition of strength of a node to the directed weighted network \vec{F}_g by defining the out-strength and in-strength of node i in \vec{F}_g as $s_i^+(g) = \sum_{j=1}^{N_g} W_{ij}(g)$ and $s_i^-(g) = \sum_{j=1}^{N_g} W_{ji}(g)$, respectively. For \vec{F}_g , we can obtain the out-strength for an arbitrary node i that entered the network at generation g_i ($g_i \geq 0$). If i was an external node when it entered the network, $s_i^+(g) = (\theta + 1)^{g-g_i}$; otherwise, if i was an internal node when it was born, $s_i^+(g) = 2(\theta + 1)^{g-g_i}$. Therefore, after each new iteration, the out-strength of a node increases by a factor of θ . It is easy to obtain that the node out-strength in \vec{F}_g obeys a distribution of power law form with the exponent being $1 + \frac{2 \ln 2}{\ln(\theta + 1)}$. Note that in some realistic networks, the node strength also display a broad distribution⁴³.

Formulation of biased walks in the weighted directed network. After introducing the construction and properties of the weighted directed network \vec{F}_g , we now define and study biased discrete-time



random walks performing \vec{F}_g . Let $r_{ij}(g) = W_{ij}(g)/s_i^+(g)$ denote the transition probability that a particle jumps from node i to its neighboring node j per time step. Note that $r_{ij}(g)$ constitutes an entry of transition matrix $R_g = (S_g)^{-1}W_g$, where S_g is the diagonal out-strength matrix of \vec{F}_g , with the i th diagonal entry of S_g being $s_i^+(g)$.

In this paper, we focus on a specific case of biased random walks, often called trapping problem, in \vec{F}_g in the presence of a trap placed at the central hub node, i.e., the internal node generated at the first iteration. To facilitate the description of the following text, all the N_g nodes in \vec{F}_g are labeled sequentially as $1, 2, \dots, N_g - 1, N_g$ as follows. For \vec{F}_1 , the newly generated internal node is labeled 1, the initial two nodes in \vec{F}_0 are labeled as 2 and 3, while the two new external nodes are labeled by 4 and 5. For each new iteration $g_i > 1$, we label consecutively the new nodes born at this iteration from $N_{g_i-1} + 1$ to N_{g_i} , while we keep the labels of those nodes created before iteration g_i unchanged.

For the trapping problem, what we are concerned with are the trapping time and the average trapping time. Let $T_i^{(g)}$ represent the trapping time for a particle initially placed at node i ($i \neq 1$) in \vec{F}_g to arrive at the trap node for the first time, which is equal to the FPT from the i to the trap. The average trapping time, $\langle T \rangle_g$, is actually the MFPT to the trap, defined as the mean of $T_i^{(g)}$ over all non-trap initial nodes in network F_g :

$$\langle T \rangle_g = \frac{1}{N_g - 1} \sum_{i=2}^{N_g} T_i^{(g)}. \quad (1)$$

Below we will show how to compute the two quantities $T_i^{(g)}$ and $\langle T \rangle_g$. For $T_i^{(g)}$, it obeys the relation

$$T_i^{(g)} = 1 + \sum_{j=2}^{N_g} r_{ij} T_j^{(g)}, \quad (2)$$

which can be recast in matrix form as:

$$T = e + \bar{R}_g T, \quad (3)$$

where $T = (T_2^{(g)}, T_3^{(g)}, \dots, T_{N_g}^{(g)})^T$ is an $(N_g - 1)$ -dimensional vector, $e = (1, 1, \dots, 1)^T$ is the $(N_g - 1)$ -dimensional vector of all ones, and \bar{R}_g is a matrix of order $N_g - 1$, which a submatrix of R_g and obtained from R_g by deleting the first row and the first column corresponding to the trap. From equation (3) we have

$$T = (I - \bar{R}_g)^{-1} e = K_g e, \quad (4)$$

where I is the $(N_g - 1) \times (N_g - 1)$ identity matrix, matrix $K_g = (I - \bar{R}_g)^{-1}$ is the fundamental matrix⁴⁴ of the addressed trapping problem. Equation (4) implies

$$T_i^{(g)} = \sum_{j=2}^{N_g} K_g(i, j), \quad (5)$$

where $K_g(i, j)$ is the ij th entry of matrix K_g , representing the expected number of visitations to node j by a particle starting from node i before being absorbed by the trap. Plugging equation (5) into equation (1) yields

$$\langle T \rangle_g = \frac{1}{N_g - 1} \sum_{i=2}^{N_g} \sum_{j=2}^{N_g} K_g(i, j). \quad (6)$$

Equation (6) indicates that the computation of MFPT $\langle T \rangle_g$ can be reduced to finding the sum of all entries of the corresponding

fundamental matrix. A disadvantage of this method is that it demands a large computational effort when the network is very large. However, equation (6) provides exact results for $\langle T \rangle_g$ that can be applied to check the results for MFPT obtained using other techniques. Next we analytically determine the closed-form expression for MFPT $\langle T \rangle_g$ using an alternative approach, the results of which are consistent with those of equation (6). It should be noted that MFPT for other deterministic scale-free unweighed networks have been previously addressed^{45–47}.

Exact solution to the MFPT $\langle T \rangle_g$. The particular selection of trap location and the specific network structure allow to determine exactly the MFPT $\langle T \rangle_g$ for arbitrary g . In order to obtain a close-form expression for $\langle T \rangle_g$, we first establish the dependence of $T_i^{(g)}$ on iteration g . For a node i in \vec{F}_g , at iteration $g + 1$, its degree doubles, increasing from $k_i(g)$ to $2k_i(g)$. All these $2k_i(g)$ neighboring nodes are created at iteration $g + 1$, among which one half are external nodes with a single degree, and the other half are internal nodes with degree 2.

We now consider the trapping problem in \vec{F}_{g+1} . Let A be the FPT for a particle starting from node i to any of its $k_i(g)$ old neighbors, that is, those nodes adjacent to i at iteration g ; let B (resp. C) be the FPT for a particle starting from any of the $k_i(g)$ internal (resp. external) neighbors of i to one of its $k_i(g)$ old neighbors. Then the FPTs obey relations:

$$\begin{cases} A = \frac{\theta}{\theta + 1} (1 + C) + \frac{1}{\theta + 1} (1 + B), \\ B = \frac{1}{2} + \frac{1}{2} (1 + A), \\ C = 1 + A. \end{cases} \quad (7)$$

Eliminating B and C in equation (7), we obtain $A = 4(\theta + 1)$. Therefore, when the network grows from iteration g to iteration $g + 1$, the FPT from any node i ($i \in \vec{F}_g$) to another node j ($j \in \vec{F}_g$) increases by a factor of $4(\theta + 1)$. Hence, $T_i^{(g+1)} = 4(\theta + 1)T_i^{(g)}$ hold for any g , which is a useful for deriving the exact expression for MFPT.

Having obtained the scaling dominating the evolution for FPTs, we continue determining the MFPT $\langle T \rangle_g$. For this purpose, we introduce two intermediary quantities for any $n \leq g$: $T_{n,\text{tot}}^{(g)} = \sum_{i \in \Lambda_n} T_i^{(g)}$ and $\bar{T}_{n,\text{tot}}^{(g)} = \sum_{i \in \bar{\Lambda}_n} T_i^{(g)}$. Then,

$$T_{g,\text{tot}}^{(g)} = T_{g-1,\text{tot}}^{(g)} + \bar{T}_{g,\text{tot}}^{(g)} = (4\theta + 4)T_{g-1,\text{tot}}^{(g-1)} + \bar{T}_{g,\text{tot}}^{(g)}. \quad (8)$$

By definition, $\langle T \rangle_g = \frac{1}{N_g - 1} T_{g,\text{tot}}^{(g)}$. To find $T_{g,\text{tot}}^{(g)}$, it is necessary to explicitly determine the quantity $\bar{T}_{g,\text{tot}}^{(g)}$. To this end, we define two additional quantities for $n \leq g$: $\bar{T}_{n,\text{int}}^{(g)} = \sum_{i \in \bar{\Lambda}_{n,\text{int}}} T_i^{(g)}$ and $\bar{T}_{n,\text{ext}}^{(g)} = \sum_{i \in \bar{\Lambda}_{n,\text{ext}}} T_i^{(g)}$. Obviously, $\bar{T}_{n,\text{tot}}^{(g)} = \bar{T}_{n,\text{int}}^{(g)} + \bar{T}_{n,\text{ext}}^{(g)}$. Thus, in order to find $\bar{T}_{g,\text{tot}}^{(g)}$, one may alternatively evaluate $\bar{T}_{n,\text{int}}^{(g)}$ and $\bar{T}_{n,\text{ext}}^{(g)}$.

We first establish the relationship between $\bar{T}_{n,\text{int}}^{(g)}$ and $\bar{T}_{n,\text{ext}}^{(g)}$. By construction (see Fig. 1), at a given generation, each edge connecting two nodes u and v will give rise three new nodes (w , x , and y) in the next generation. The two external nodes x and y are separately attached to u and v , while the only internal node w is linked simultaneously to u and v . For any iteration g , the FPTs for the three new nodes satisfy: $T_x^{(g)} = 1 + T_u^{(g)}$, $T_y^{(g)} = 1 + T_v^{(g)}$, and $T_w^{(g)} = \frac{1}{2} [1 + T_u^{(g)}] + \frac{1}{2} [1 + T_v^{(g)}]$. Therefore, $T_x^{(g)} + T_y^{(g)} = 2T_w^{(g)}$. Summing this relation over all old edges at the generation before growth, we find that for all $n \leq g$, $\bar{T}_{n,\text{ext}}^{(g)} = 2\bar{T}_{n,\text{int}}^{(g)}$ always holds. In



this way, the issue of determining $\bar{T}_{g,\text{tot}}^{(g)}$ is reduced to finding $\bar{T}_{g,\text{ext}}^{(g)}$ that can be obtained as follows.

For an arbitrary external node i_{ext} in \vec{F}_g , which is created at generation g and attached to an old node i , we have $T_{i_{\text{ext}}}^{(g)} = 1 + T_i^{(g)}$, a relation valid for any node pair containing an old node and one of its new external adjacent nodes. By applying relation $T_{i_{\text{ext}}}^{(g)} = 1 + T_i^{(g)}$ to two sum (the first one is over a given old node and all its new adjacent external nodes, the other is summing the first one over all old nodes), we obtain

$$\begin{aligned} \bar{T}_{g,\text{ext}}^{(g)} &= |\bar{\Lambda}_{g,\text{ext}}| + \sum_{i \in \Lambda_{g-1}} \left(k_i(g-1) \times T_i^{(g)} \right) \\ &= |\bar{\Lambda}_{g,\text{ext}}| + \left(\bar{T}_{g-1,\text{ext}}^{(g)} + 2\bar{T}_{g-1,\text{int}}^{(g)} \right) \\ &\quad + \left(2\bar{T}_{g-2,\text{ext}}^{(g)} + 4\bar{T}_{g-2,\text{int}}^{(g)} \right) + \dots \\ &\quad + \left(2^{g-2}\bar{T}_{1,\text{ext}}^{(g)} + 2^{g-1}\bar{T}_{1,\text{int}}^{(g)} \right) \\ &= 2 \times 4^{g-1} + 2\bar{T}_{g-1,\text{ext}}^{(g)} + 4\bar{T}_{g-2,\text{ext}}^{(g)} + \dots + 2^{g-1}\bar{T}_{1,\text{ext}}^{(g)} \end{aligned} \quad (9)$$

From equation (9), one can derive the recursive relation

$$\bar{T}_{g+1,\text{ext}}^{(g+1)} = 16(\theta+1)\bar{T}_{g,\text{ext}}^{(g)} - (4\theta+2)4^g. \quad (10)$$

Considering the initial condition $\bar{T}_{2,\text{ext}}^{(2)} = 48\theta^2 + 80\theta + 40$, equation (10) is solved to yield

$$\bar{T}_{g,\text{ext}}^{(g)} = \frac{12\theta^2 + 17\theta + 7}{(\theta+1)(4\theta+3)} 2^{4g-4}(\theta+1)^g + \frac{2\theta+1}{4\theta+3} 2^{2g-1}. \quad (11)$$

Because $\bar{T}_{g,\text{tot}}^{(g)} = \bar{T}_{g,\text{int}}^{(g)} + \bar{T}_{g,\text{ext}}^{(g)}$ and $\bar{T}_{g,\text{ext}}^{(g)} = 2\bar{T}_{g,\text{int}}^{(g)}$, we have

$$\bar{T}_{g,\text{tot}}^{(g)} = \frac{36\theta^2 + 51\theta + 21}{(\theta+1)(4\theta+3)} 2^{4g-5}(\theta+1)^g + \frac{6\theta+3}{4\theta+3} 2^{2g-2}. \quad (12)$$

Inserting equation (12) into equation (8) leads to

$$\begin{aligned} T_{g,\text{tot}}^{(g)} &= (4\theta+4)T_{g-1,\text{tot}}^{(g-1)} + \frac{6\theta+3}{4\theta+3} 2^{2g-2} \\ &\quad + \frac{36\theta^2 + 51\theta + 21}{(\theta+1)(4\theta+3)} 2^{4g-5}(\theta+1)^g. \end{aligned} \quad (13)$$

Using $T_{1,\text{tot}}^{(1)} = 8\theta + 6$, equation (13) is solved to get

$$\begin{aligned} T_{g,\text{tot}}^{(g)} &= \frac{12\theta^3 + 17\theta^2 + 7\theta}{\theta(\theta+1)(4\theta+3)} 2^{4g-3}(\theta+1)^g \\ &\quad + \frac{16\theta^3 + 28\theta^2 + 20\theta + 6}{\theta(\theta+1)(4\theta+3)} 2^{2g-3}(\theta+1)^g \\ &\quad - \frac{3(\theta+1)(2\theta+1)}{\theta(\theta+1)(4\theta+3)} 2^{2g-2}. \end{aligned} \quad (14)$$

Then, the rigorous expression for the MFPT $\langle T \rangle_g$ of the weighted directed network \vec{F}_g is

$$\begin{aligned} \langle T \rangle_g &= \frac{12\theta^3 + 17\theta^2 + 7\theta}{8\theta(\theta+1)(4\theta+3)} 2^{2g}(\theta+1)^g \\ &\quad + \frac{16\theta^3 + 28\theta^2 + 20\theta + 6}{8\theta(\theta+1)(4\theta+3)} (\theta+1)^g - \frac{3(\theta+1)(2\theta+1)}{4\theta(\theta+1)(4\theta+3)}. \end{aligned} \quad (15)$$

We have checked the analytical solution in equation (15) against extensive numerical results obtained from equation (6), see Fig. 3. For different θ and g , both the analytical and numerical results are in full agreement with each other, indicating that the explicit expression

in equation (15) is correct. In addition, for the particular case $\theta = 1$, the network \vec{F}_g is reduced to F_g , and equation (15) recovers the result²³ previously obtained for F_g . This also validates equation (15).

We proceed to express $\langle T \rangle_g$ in terms of the network size N_g in order to uncover how $\langle T \rangle_g$ scales with N_g . From $N_g = 4^g + 1$, we have $g = \log_4(N_g - 1)$. Then,

$$\begin{aligned} \langle T \rangle_g &= \frac{12\theta^3 + 17\theta^2 + 7\theta}{8\theta(\theta+1)(4\theta+3)} (N_g - 1)^{1 + \log_4(\theta+1)} \\ &\quad + \frac{16\theta^3 + 28\theta^2 + 20\theta + 6}{8\theta(\theta+1)(4\theta+3)} (N_g - 1)^{\log_4(\theta+1)} \\ &\quad - \frac{3(\theta+1)(2\theta+1)}{4\theta(\theta+1)(4\theta+3)}. \end{aligned} \quad (16)$$

For a very large network (i.e., $N_g \rightarrow \infty$), the leading term of $\langle T \rangle_g$ can be represented as:

$$\langle T \rangle_g \sim (N_g)^{1 + \log_4(\theta+1)}. \quad (17)$$

Equation (17) shows that for the directed weighted network \vec{F}_g , the MFPT $\langle T \rangle_g$ behaves as a power-law function of the network size N_g , with the exponent $\eta(\theta) = 1 + \log_4(\theta+1)$ increasing with the weight parameter θ . Thus, the weight reciprocity has an essential effect on the efficiency of the trapping problem, measured by the MFPT.

Eigenvalues of the fundamental matrix. We now study the eigenvalues of the fundamental matrix K_n of the trapping problem addressed above. We will determine all the eigenvalues of the fundamental matrix as well as their multiplicities. Moreover, we will show that the largest eigenvalue has the same leading scaling as that of $\langle T \rangle_g$. To attain this goal, we introduce matrix P_g defined by $P_g = K_g^{-1}$. Let $\lambda_i(g)$ and $\sigma_i(g)$, where $i = 1, 2, \dots, N_n - 1$, denote the eigenvalues of P_g and K_g , such that $\lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \dots \leq \lambda_{N_g-1}(g)$ and $\sigma_1(g) \geq \sigma_2(g) \geq \sigma_3(g) \geq \dots \geq \sigma_{N_n-1}(g)$. Then, the one-to-one relation $\lambda_i(g) = 1/\sigma_i(g)$ holds. Thus, to compute the eigenvalues of matrix K_n , we can alternatively determine the eigenvalues for P_g . In the sequel, we will use the decimation method^{48,49} to find all the eigenvalues of matrix P_g .

Full spectrum of fundamental matrix. The decimation procedure^{48,49} makes it possible to obtain the eigenvalues for related matrix of the current iteration from those of the previous iteration.

We now consider the eigenvalue problem for matrix P_{g+1} . Let α denote the set of nodes in network \vec{F}_g , and β the set of nodes created at iteration $g+1$. Suppose that $\lambda_i(g+1)$ is an eigenvalue of P_{g+1} , and $u = (u_\alpha, u_\beta)^\top$ is an eigenvector associated with $\lambda_i(g+1)$, where u_α and u_β correspond to nodes belonging to sets α and β , respectively. Then, eigenvalue equation for matrix P_{g+1} can be represented in a block form:

$$\begin{bmatrix} P_{\alpha,\alpha} & P_{\alpha,\beta} \\ P_{\beta,\alpha} & P_{\beta,\beta} \end{bmatrix} \begin{bmatrix} u_\alpha \\ u_\beta \end{bmatrix} = \lambda_i(g+1) \begin{bmatrix} u_\alpha \\ u_\beta \end{bmatrix}, \quad (18)$$

where $P_{\alpha,\alpha}$ and $P_{\beta,\beta}$ are the identity matrix.

Equation (18) can be expressed as two equations:

$$P_{\alpha,\alpha}u_\alpha + P_{\alpha,\beta}u_\beta = \lambda_i(g+1)u_\alpha, \quad (19)$$

$$P_{\beta,\alpha}u_\alpha + P_{\beta,\beta}u_\beta = \lambda_i(g+1)u_\beta. \quad (20)$$

Equation (20) implies

$$u_\beta = \frac{1}{\lambda_i(g+1) - 1} P_{\beta,\alpha}u_\alpha, \quad (21)$$



provided that $\lambda_i(g + 1) \neq 1$. Inserting equation (21) into equation (19) yields

$$P_{\alpha,\beta} P_{\beta,\alpha} u_\alpha = [\lambda_i(g + 1) - 1]^2 u_\alpha. \quad (22)$$

In this way, we reduce the problem of determining the eigenvalue $\lambda_i(g + 1)$ for matrix P_{g+1} of order 4^{g+1} to finding the eigenvalue problem of matrix $P_{\alpha,\beta} P_{\beta,\alpha}$ with a smaller order 4^g .

We can prove (see Methods) that

$$P_{\alpha,\beta} P_{\beta,\alpha} = I_g - \frac{1}{2\theta + 2} P_g, \quad (23)$$

where I_g is the identity matrix of order 4^g , identical to that of P_g . Equation (23) relates the product matrix $P_{\alpha,\beta} P_{\beta,\alpha}$ to matrix P_g . Therefore, the eigenvalues of matrix P_{g+1} can be expressed in terms of those of matrix P_g .

We next show how to obtain the eigenvalues of P_{g+1} through the eigenvalues of P_g . According to equations (22) and (23), we can derive

$$P_g u_\alpha = - (2\theta + 2) [\lambda_i^2(g + 1) - 2\lambda_i(g + 1)] u_\alpha. \quad (24)$$

Hence, if $\lambda_i(g)$ is an eigenvalue of P_g associated with eigenvector u_α , equation (24) indicates

$$\lambda_i(g) = - (2\theta + 2) [\lambda_i(g + 1)^2 - 2\lambda_i(g + 1)]. \quad (25)$$

Solving the above quadratic equation in the variable $\lambda_i(g + 1)$ given by equation (25), one obtains the two roots:

$$\lambda_{i,1}(g + 1) = 1 - \sqrt{1 - \frac{\lambda_i(g)}{2\theta + 2}}, \quad (26)$$

and

$$\lambda_{i,2}(g + 1) = 1 + \sqrt{1 - \frac{\lambda_i(g)}{2\theta + 2}}. \quad (27)$$

Equations (26) and (27) relate $\lambda_i(g + 1)$ to $\lambda_i(g)$, with each eigenvalue $\lambda_i(g)$ of P_g giving rise two different eigenvalues of P_{g+1} . As a matter of fact, all eigenvalues of the P_{g+1} can be obtained by these two recursive relations. In Methods, we determine the multiplicity of each eigenvalue and show that all the eigenvalues can be found by equations (26) and (27).

Since there is a one-to-one relation between the eigenvalues of P_g and the fundamental matrix K_g , we thus have also found all the eigenvalues of K_g .

The largest eigenvalue of fundamental matrix and MFPT. In the above, we have determined all eigenvalues for the inverse P_g of the fundamental matrix K_g and thus all eigenvalues of K_g . Here we continue to estimate the greatest eigenvalue $\sigma_{\max}(g)$ of the fundamental matrix K_g , which actually equals the reciprocal of the smallest eigenvalue for matrix P_g denoted by $\lambda_{\min}(g)$. Below we will show that in a large network the leading behavior of the MFPT $\langle T \rangle_g$ for trapping in \vec{F}_g and the reciprocal of $\lambda_{\min}(g)$ is identical, that is, $\langle T \rangle_g \sim 1/\lambda_{\min}(g) = \sigma_{\max}(g)$.

We begin by providing some useful properties of eigenvalues for matrix P_g . Assume that Δ_g is the set of the 4^g eigenvalues of matrix P_g , namely, $\Delta_g = \{\lambda_1(g), \lambda_2(g), \lambda_3(g), \dots, \lambda_{4^g}(g)\}$. According to the above analysis, Δ_g can be categorized into two subsets $\Delta_g^{(1)}$ and $\Delta_g^{(2)}$ satisfying $\Delta_g = \Delta_g^{(1)} \cup \Delta_g^{(2)}$, where $\Delta_g^{(1)}$ consists of all eigenvalues 1, while $\Delta_g^{(2)}$ contains the rest eigenvalues. Thus,

$$\Delta_g^{(1)} = \underbrace{\{1, 1, 1, \dots, 1, 1\}}_{2 \times 4^{g-1}}. \quad (28)$$

These $2 \times 4^{g-1}$ eigenvalues are labeled sequentially by $\lambda_{4^{g-1}+1}(g), \lambda_{4^{g-1}+2}(g), \dots, \lambda_{3 \times 4^{g-1}}(g)$, since they provide a natural increasing order of all eigenvalues for P_g , as will be shown.

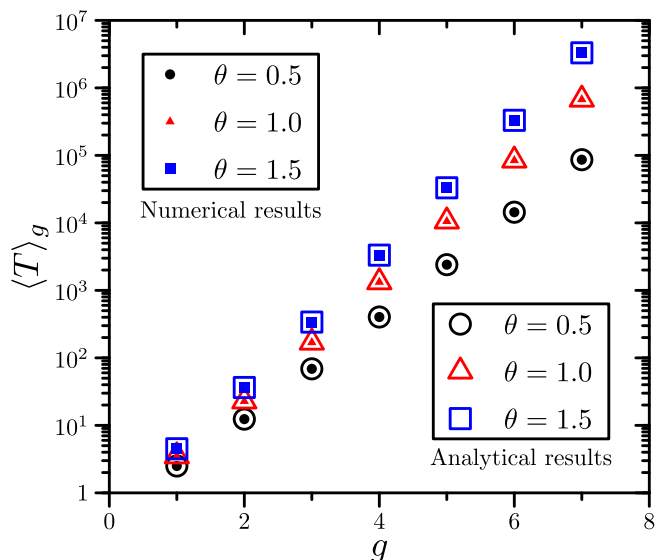


Figure 3 | MFPT $\langle T \rangle_g$ as a function of g for different networks with various θ . The filled symbols are the data obtained by direct calculation from equation (6); while the empty symbols are those exact analytical values given by equation (15).

The remaining $2 \times 4^{g-1}$ eigenvalues in set $\Delta_g^{(2)}$ are all determined by equations (26) and (27). Let $\lambda_1(g-1), \lambda_2(g-1), \lambda_3(g-1), \dots, \lambda_{4^{g-1}}(g-1)$ be the 4^{g-1} eigenvalues of matrix P_{g-1} , arranged in an increasing order $\lambda_1(g-1) \leq \lambda_2(g-1) \leq \lambda_3(g-1) \leq \dots \leq \lambda_{4^{g-1}}(g-1)$. Then, for each eigenvalue $\lambda_i(g-1)$ of P_{g-1} , equations (26) and (27) produce two eigenvalues of P_g , which are labeled as $\lambda_i(g)$ and $\lambda_{4^g-i+1}(g)$:

$$\lambda_i(g) = 1 - \sqrt{1 - \frac{\lambda_i(g-1)}{2\theta + 2}} \quad (29)$$

and

$$\lambda_{4^g-i+1}(g) = 1 + \sqrt{1 - \frac{\lambda_i(g-1)}{2\theta + 2}}. \quad (30)$$

Plugging each eigenvalue of P_{g-1} into equations (26) and (27) generates all eigenvalues in $\Delta_g^{(2)}$.

It is easy to see that $\lambda_i(g)$ given by equation (29) monotonously increases with $\lambda_i(g-1)$ and belongs to interval $(0, 1)$, while $\lambda_{4^g-i+1}(g)$ provided by equation (30) monotonously decreases with $\lambda_i(g-1)$ and lies in interval $(1, 2)$. Thus, $\lambda_1(g), \lambda_2(g), \lambda_3(g), \dots, \lambda_{4^g}(g)$ provide an increasing order of all eigenvalues for matrix P_g .

We hasten to estimate $\lambda_{\min}(g)$ of matrix P_g . From the above arguments, the smallest eigenvalue $\lambda_{\min}(g)$ must be the one generated from $\lambda_{\min}(g-1)$ through equation (29):

$$\lambda_{\min}(g) = 1 - \sqrt{1 - \frac{\lambda_{\min}(g-1)}{2\theta + 2}}. \quad (31)$$

Using Taylor's formula, we have

$$\lambda_{\min}(g) \approx 1 - \left[1 - \frac{\lambda_{\min}(g-1)}{4\theta + 4} \right] = \frac{\lambda_{\min}(g-1)}{4\theta + 4}. \quad (32)$$

Considering $\lambda_{\min}(1) = 1 - \sqrt{1 - \frac{1}{\theta + 1}}$, equation (32) is solved to yield

$$\lambda_{\min}(g) \approx \left(1 - \sqrt{1 - \frac{1}{\theta + 1}} \right) (4\theta + 4)^{1-g}. \quad (33)$$



Thus,

$$\frac{1}{\lambda_{\min}(g)} \approx \frac{1}{4(\theta+1)^2} \left(1 + \sqrt{1 - \frac{1}{\theta+1}} \right) 4^g (\theta+1)^g, \quad (34)$$

which, together with equation (15), means that $\frac{1}{\lambda_{\min}(g)}$ and $\langle T \rangle_g$ have the same dominating term and thus identical leading scaling.

Discussion

Real-life weighted networks exhibit a rich and diverse reciprocity structure. In this paper, we have proposed a scale-free weighted directed network with asymmetric edge weights, which are controlled by a parameter characterizing the network reciprocity. We then studied random walks performed on the network with a trap fixed at the central hub node. Applying two different approaches, we have evaluated the MFPT to the trap. Moreover, based on the self-similar architecture of the network, we have found all the eigenvalues and their multiplicities of the fundamental matrix describing the random-walk process, the largest one of which has the same leading scaling as that of the MFPT. The obtained results indicate that the MFPT scales as a power-law function of the system size, with the power exponent increasing with the weight parameter, revealing that the reciprocity has a significant impact on dynamical processes running on weighted networks. Finally, it should be mentioned that although we have only studied a particular network, our methods can also be applied to other self-similar networks, obtaining analogous results. Thus, our work deepens the understanding of random-walk dynamics in complex systems and opens a novel avenue to control random walks in a weighted network by changing its reciprocity.

Methods

Proof of equation (23). In order to prove equation (23), we rewrite $P_{\alpha,\beta}$ and $P_{\beta,\alpha}$ in the block form as

$$P_{\alpha,\beta} = (U_1, U_2, \dots, U_{E_g}) \quad (35)$$

and

$$P_{\beta,\alpha} = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_{E_g} \end{pmatrix}, \quad (36)$$

respectively. In equations (35) and (36), $E_g = 4^g$ is the number of edges in F_g ; U_i ($1 \leq i \leq E_g$) is a $4^g \times 3$ matrix describing the transition probability from the 4^g non-trap nodes of F_g to the three nodes newly generated by the i th edge of F_g ; similarly, V_i ($1 \leq i \leq E_g$) is a 3×4^g matrix indicating the transition probability from the three new nodes created by the i th edge to those 4^g old non-trap nodes belonging to F_g . Then,

$$\begin{aligned} P_{\alpha,\beta} P_{\beta,\alpha} &= \sum_{i=1}^{E_g} U_i V_i \\ &= \sum_{i=1}^{E_g} \left(\frac{a_i}{\theta+1} e_{li} + \frac{b_i}{\theta+1} e_{ri}, \frac{\theta}{\theta+1} a_i e_{li}, \frac{\theta}{\theta+1} b_i e_{ri} \right) \times \\ &\quad \begin{pmatrix} -\frac{e_{li}^\top + e_{ri}^\top}{2} \\ -e_{li}^\top \\ -e_{ri}^\top \end{pmatrix} \\ &= -\frac{1}{2\theta+2} \times \\ &\quad \sum_{i=1}^{E_g} \left[(2\theta+1) (a_i e_{li} e_{li}^\top + b_i e_{ri} e_{ri}^\top) + a_i e_{li} e_{ri}^\top + b_i e_{ri} e_{li}^\top \right] \\ &= I_g - \frac{1}{2\theta+2} P_g, \end{aligned} \quad (37)$$

which completes the proof of equation (23). Note that in equation (37), l_i and r_i are the two endpoints of the i th edge of F_g ; e_i is a vector having only one nonzero element 1 at

i th entry with other entries being zeros; a_i and b_i are two entries of P_g corresponding to edges (l_i, r_i) and (r_i, l_i) , respectively.

Alternative proof of equation (23). Equation (23) can also be proved using another technique. Assume that $R_g = P_{\alpha,\beta} P_{\beta,\alpha}$ and $Q_g = I_g - \frac{1}{2\theta+2} P_g$. In order to prove

$P_{\alpha,\beta} P_{\beta,\alpha} = I_g - \frac{1}{2\theta+2} P_g$, it suffices to show that the entries of R_g are equal to their counterparts of Q_g . For matrix Q_g , it is easy to see that its entries are: $Q_g(i,i) = \frac{2\theta+1}{2\theta+2}$ for $i = j$ and $Q_g(i,j) = -\frac{1}{2\theta+2} P_g(i,j)$ otherwise. If $P_{g+1}(i,j)$ denotes the (i,j) entry of matrix P_{g+1} , the entries of $R_g(i,j)$ of matrix R_g can be evaluated by distinguishing two cases: $i = j$ and $i \neq j$.

For the case of $i = j$, the diagonal element of R_g is

$$\begin{aligned} R_g(i,i) &= \\ &= \sum_{z \in \beta} P_{g+1}(i,z) P_{g+1}(z,i) = \sum_{z \in \beta} \frac{W_{iz}(g+1)}{s_i^+(g+1)} \frac{W_{zi}(g+1)}{s_z^+(g+1)} \\ &= \frac{1}{2} \sum_{\substack{z \in \beta, i \sim z \\ k_i(g+1)=2}} \frac{W_{iz}(g+1)}{s_i^+(g+1)} + \sum_{\substack{z \in \beta, i \sim z \\ k_i(g+1)=1}} \frac{W_{iz}(g+1)}{s_i^+(g+1)} \\ &= \frac{1}{2} \frac{s_i^+(g)}{s_i^+(g+1)} + \frac{\theta s_i^+(g)}{s_i^+(g+1)} \\ &= \frac{2\theta+1}{2\theta+2} = Q_g(i,i), \end{aligned} \quad (38)$$

where the relation $s_i^+(g+1) = (\theta+1)s_i^+(g)$ is used. In equation (38), $i \sim z$ indicates that two nodes i and z are adjacent in network F_{g+1} .

For the other case of $i \neq j$, the non-diagonal element of R_g is

$$\begin{aligned} R_g(i,j) &= \sum_{z \in \beta} P_{g+1}(i,z) P_{g+1}(z,j) \\ &= \sum_{\substack{A_{g+1}(i,z)=1 \\ A_{g+1}(z,j)=1}} \frac{W_{iz}(g+1)}{s_i^+(g+1)} \frac{W_{zj}(g+1)}{s_z^+(g+1)} \\ &= \frac{1}{2} \frac{W_{ij}(g)}{s_i^+(g+1)} = -\frac{1}{2\theta+2} P_g(i,j) \\ &= Q_g(i,j), \end{aligned} \quad (39)$$

which, together with (38) proves equation (23).

Multiplicities of eigenvalues. By numerically computing the eigenvalues for the first several iterations, we can observe some important phenomena and properties about

the structure of the eigenvalues. When $g = 1$, the eigenvalues of P_1 are $1 - \sqrt{1 - \frac{1}{\theta+1}}$

and $1 + \sqrt{1 - \frac{1}{\theta+1}}$, both of which have a multiplicity of 2. When $g = 2$, P_2 have 16 eigenvalues: eigenvalue 1 with degeneracy 8 and 4 two-fold other eigenvalues

generated by $1 - \sqrt{1 - \frac{1}{\theta+1}}$ and $1 + \sqrt{1 - \frac{1}{\theta+1}}$. When $g \geq 3$, all the eigenvalues P_g

can be put into two classes. The first class includes eigenvalue 1 and those generated by 1, which display the following feature that each eigenvalue appearing at a given iteration g_i will continue to appear at all subsequent generations greater than g_i . The second class contains those eigenvalues generated by the two eigenvalues

$1 - \sqrt{1 - \frac{1}{\theta+1}}$ and $1 + \sqrt{1 - \frac{1}{\theta+1}}$ of P_1 . Each eigenvalue in this class is two-fold,

and each eigenvalue of a given iteration g_i does not appear at any of subsequent iterations larger than g_i . For the two eigenvalue classes, each eigenvalue (other than 1) of current generation keeps the multiplicity of its father of the previous generation.

Using the above-observed properties of the eigenvalue structure, we can determine the multiplicities of all eigenvalues. Let $M_g(\lambda)$ denote the multiplicity of eigenvalue λ of matrix P_g . We first determine the number of eigenvalue 1 for P_g . To this end, let $r(X)$ denote the rank of matrix X . Then

$$M_g(\lambda=1) = 4^g - r(P_g - 1 \times I_g). \quad (40)$$

For $g = 1$, $M_1(1) = 0$; for $g = 2$, $M_2(1) = 8$. For $g \geq 2$, it is obvious that $r(P_{g+1} - I_{g+1}) = r(P_{\alpha,\beta}) + r(P_{\beta,\alpha})$, where $r(P_{\alpha,\beta})$ and $r(P_{\beta,\alpha})$ can be determined in the following way.

We first show that $P_{\beta,\alpha}$ is a full column rank matrix. Let

$$\phi = (\phi_1, \phi_2, \dots, \phi_{3 \times 4^g})^\top = \sum_{\substack{i \in \alpha \\ i \neq 1}} k_i M_i, \quad (41)$$

where M_i is the column vector of $P_{\beta,\alpha}$ representing the i th column of $P_{\beta,\alpha}$. Let

$M_i = (M_{1,i}, M_{2,i}, \dots, M_{3 \times 4^g, i})^\top$. Suppose that $\phi = 0$. Then, we can prove that for an arbitrary k_i , $k_i = 0$ always holds. By construction, for any node $i \in \alpha$, there exists a



new leaf node $l \in \beta$ attached to i . Then, for $\phi_l = k_1 M_{1,l} + k_2 M_{2,l} + \dots + k_{3 \times 4^g} M_{3 \times 4^g, l}$, only $M_{i,l} \neq 0$ but all $M_{x,l} = 0$ for $x \neq i$. From $\phi_l = 0$, we have $k_i = 0$. Therefore, $r(P_{\beta, \alpha}) = 4^g$. Analogously, we can verify that $P_{\alpha, \beta}$ is a full row rank matrix and $r(P_{\alpha, \beta}) = 4^g$. Combining the above results, the multiplicity of eigenvalue 1 of P_g is

$$M_g(\lambda=1) = \begin{cases} 0, & g=1, \\ 2 \times 4^{g-1}, & g \geq 2. \end{cases} \quad (42)$$

We proceed to compute the multiplicities of other eigenvalues generated by 1 that are in the first eigenvalue class. Since every eigenvalue at a given iteration keeps the multiplicity of its father at the preceding iteration, for matrix P_g , the multiplicity of each first-generation descendant of eigenvalue 1 is $2 \times 4^{g-2}$, the multiplicity of each second-generation descendant of eigenvalue 1 is $2 \times 4^{g-3}$, and the multiplicity of each $(g-2)$ nd generation descendant of eigenvalue 1 is 2×4 . Moreover, we can derive that the number of the i th ($0 \leq i \leq g-2$) generation distinct descendants of eigenvalue is 2^i , where 0th generation descendants refer to the $2 \times 4^{g-1}$ eigenvalues 1 themselves. Finally, it is easy to verify that the number of all the eigenvalues in the second eigenvalue class is $4 \times 2^{g-1}$. Hence, the total number of eigenvalues of matrix P_g is

$$\sum_{i=0}^{g-2} [(2 \times 4^{g-1-i}) \times 2^i] + 4 \times 2^{g-1} = 4^g, \quad (43)$$

indicating that all the eigenvalues of P_g are successfully found.

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Author contributions

Z.Z.Z. designed the research; H.L. and Y.B.S. performed the research; Z.Z.Z. and H.L. wrote the manuscript.

Additional information

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