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Renormalization and small-world model of fractal quantum repeater networks

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Quantum networks provide access to exchange of quantum information. The primary task of quantum networks is to distribute entanglement between remote nodes. Although quantum repeater protocol enables long distance entanglement distribution, it has been restricted to one-dimensional linear network. Here we develop a general framework that allows application of quantum repeater protocol to arbitrary quantum repeater networks with fractal structure. Entanglement distribution across such networks is mapped to renormalization. Furthermore, we demonstrate that logarithmical times of recursive such renormalization transformations can trigger fractal to small-world transition, where a scalable quantum small-world network is achieved. Our result provides new insight into quantum repeater theory towards realistic construction of large-scale quantum networks.

By exploiting the probabilistic prediction nature of quantum mechanics and the nonlocal correlation of entanglement[1,2], new technology of quantum communication has been developed[3]. For example, quantum teleportation allows faithful teleportation of unknown quantum states, and quantum cryptography (Ekert91 protocol) enables truly secure communication[3]. Quantum nodes can store and manipulate photons locally, and their interconnection via quantum channel, e.g., optical fiber, gives birth to quantum networks. Quantum networks are backbone of quantum communication and distributed quantum computing[4]. A prototype of quantum network has been reported recently[5]. Quantum networks can be seen as large and complex system of quantum states. How to characterize and understand such quantum system remains a challenge. It is no longer effectively described by a global density operator ρ[4]. However, complex networks which describe a wide range of natural and social system[6–8], provide conceptual basis for in-depth investigation of topological properties of quantum networks. It involves exciting phenomena. For instance, entanglement percolation[9–11] and the peculiar behavior of quantum random networks[12] have received much attention. What's more, it has opened new perspective for study of entanglements: map complex networks into entangled states, and vice versa[13,14].

As essential ingredients for quantum information, however, entanglement is such fragile resource that suffer fatal photon loss, decoherence caused by noise, and imperfection of quantum local operations[3,15–17]. In consequence, the state fidelity or degree of entanglement decreases exponentially with channel length. Quantum repeater protocols (QRP) are one of most promising solutions designed to tackle such challenging problem[3,15–20]. The general principle of QRP is illustrated in Fig. 1(a)–(c). In principle, QRP allow one to establish long-distance entanglement with fidelity close to unity, while the required time increases, e.g., polynomially with channel length (it depends on specific QRP). Thereby quantum repeaters hold promise for building large-scale quantum networks. Then a fundamental problem comes to us: what's the possible topology of quantum repeater network (QRN), and how to perform QRP on such network?

In this work, we address this problem, with focus on the interplay between entanglement distribution and topology of quantum networks. A practical scenario of entanglement distribution has to consider the complex topology of quantum networks. Recent studies suggest that the topology of quantum networks strongly affects their performance[9–11]. How well can we say about the topology of QRN? It's a problem that has not been seriously considered. Motivated by the rich and intriguing topology of complex networks, we envisage the topology of QRN as follows. The exponential decay of fidelity requires that QRN be fractal. And scale-free network is a plausible option for QRN[6,7,21]. Moreover, as a generic characteristic of real-world networks, small-world effect[6,7,22] is able to reinforce the scalability of quantum networks[11]. We combine these elements with QRN. Next, so far, QRP have been elucidated for one-dimensional linear network. In order to apply QRP to fractal QRN, we draw on concepts and methodology from statistical physics. We find a clue to relate implementation of QRP to renormalization transformation. As a result, entanglement distribution over arbitrary fractal QRN corresponds to a process of successive renormalization transformations.
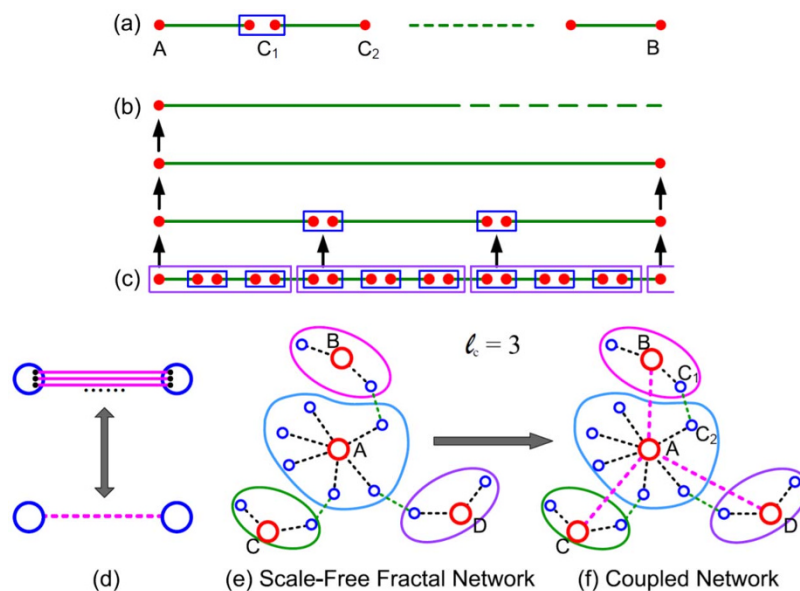
Figure 1 | Renormalization and its relationship with quantum repeaters. Top three panels (also see Ref. 16): principle of quantum repeaters. (a) The entire channel is divided into N auxiliary segments, avoiding exponential attenuation of fidelity. Entanglements are repeatedly created for each segments. (b) and (c) Nested purification that combines entanglement swapping and purification are successively performed in a hierarchical way. Adjacent segments are connected and extended to longer distance, eventually two remote nodes are connected via perfect entanglements. Bottom panel: schematic illustration of coupled renormalization. (d) Quantum node (circle) is a composite system composed of ensembles of qubits (dots). A pair of quantum nodes is connected by multiple copies of bipartite entangled states (solid lines) . For simplicity, they are represented by single dashed lines in Fig. 1(e) and (f). (e) Nodes are assigned to different boxes according to the MEMB algorithm. (f) Single level CR is performed.

Results

Relationship between QRP and renormalization transformation. Here we offer a new perspective on QRP. As shown in Fig. 1(a)–(c), the operations of QRP exhibit a hierarchical characteristic and self-similarly nested structure. Using standard box-counting method, it's easy to compute fractal dimension $d_B = 1$. Historically Kadanoff's seminal picture that spin blocks hierarchically nest in a self-similar manner basically stimulated Wilson's renormalization group theory, which is renowned as a powerful tool to the problem of phase transition[23,24]. Here we point out that the implementation of QRP with nested structure actually corresponds to renormalization process in network setting.

As shown in Fig. 1(a)–(c), the quantum channel between nodes A and B is divided into N segments, and every ℓ_c consecutive segments are grouped together into a unit which is extended to larger length-scale. This procedure is repeated with n nesting levels, where $N = \ell_c^{n3,15,16}$. If we interpret ℓ_c as a parameter which is none other than the transforming length-scale, then above procedure can be viewed as real space renormalization transformation. Guided by this remarkable idea, we present an universal framework which allows one to apply QRP to arbitrary QRN with fractal structure.

We find clear correspondences between one-dimensional and high dimensional quantum repeaters. Undoubtedly, the exponential decay of fidelity imposes strong constraint on the way nodes are interconnected. Basically nodes are interconnected in accordance with local attachment: a node prefers to link to neighboring nodes via intermediate repeater nodes, rather than distant nodes. This connection fashion gives rise to fractal structure[25]. Then fractal QRN can be substituted for the 1D linear network. And the length of 1D chain is replaced by the diameter of underneath fractal QRN

$$D_0 \sim N^{1/d_B}, \tag{1}$$

namely, largest distance between nodes (in graphic sense, the distance between two nodes is defined as the number of links along the shortest path[6,7]). Compared with segmentation fashion

of 1D chain, the entire network is divided into boxes, whose size is equal to ℓ_c, see Fig. 1(e). It will be desirable if the nesting level takes a similar form:

$$N^{1/d_B} = \ell_c^{n_c}. \tag{2}$$

We will prove that it is indeed the case. It answers an intuitive problem: how many nesting levels n_c are required for quantum networks with N nodes? Also we will show that it implies a structural transition where small-world is obtained.

Renormalization was successfully introduced into complex networks by Song *et al.*, uncovering the self-similarity of complex networks[26]. A network is renormalized according to the box-covering technique[26] (see Fig. 1(e)–(f)). The basic idea is as follows: tile the entire network with minimum number of boxes N_B, where the distance between nodes within any box is smaller than the box size, namely, the transforming length-scale ℓ_B. Each box is then replaced by a supernode. These supernodes are connected if there is at least one link between nodes in their respective boxes. It defines the fractal dimension d_B in terms of a power law:

$$\frac{N_B}{N} \sim \ell_B^{-d_B}. \tag{3}$$

Apply this transformation R_{ℓ_B} to a fractal network G_0, which is scale invariant, then we have $R_{\ell_B}(G_0) = G_0$.

Several algorithms[28,29] have been proposed to coarse-grain complex networks, nevertheless, not all of them are useful here. We choose the MEMB algorithm to divide the entire network into boxes[29]. It's a geometric algorithm which has the advantages of guaranteeing connectivity within boxes, isolating hubs of different boxes and avoiding overlap between boxes.

We then introduce some modifications. For clarity, this is rephrased in network language. We select the hub of a box (most connected node) as representative node, playing the role of supernode. If there is one link between two boxes, then add one link connecting their hubs. Otherwise, no links are attached. Above presentation instructs us to create shortcuts (long-distance entangled state with

high fidelity) between which pair of representative nodes. As shown in Fig. 1(f), the corresponding shortcuts denoted by pink links form a coarse-grained network (CGN). As a result, the CGN is reconstructed and coupled to the initial network. Because of the self-similarity of underling network, the resulting CGN is topologically equivalent to the original one, and above renormalization processing can be iteratively applied to previous CGN with fixed ℓ_c, and so on until the critical nesting level (see Eq. (8)). Notice that the requirement of local attachment is fulfilled throughout entire process. Eventually, the superposition of each level CGN forms a new network, namely, the coupled network (CN). For the sake of distinction between the standard and this modified renormalization, we name the later coupled renormalization (CR).

We have generalize quantum repeaters to arbitrary high but finite fractal dimension, in the sense that when $d_B = 1$, CR is automatically reduced to 1D quantum repeaters. One possible application of CR for 1D linear network is shown in Fig. 1(c). In 1D case, each representative node is merely connected to two representative nodes (nodes below arrows). In regard to fractal QRN, however, a representative node (red circles) has probability $P(k')$ to link k' representative nodes (it depends on the degree distribution) via the short path composed of entangled links between two boxes (green links). For instance, in Fig. 1(f), hubs A and B are connected through a path B-C_1-C_2-A, whose length is $\ell_c = 3$. Each of such a path corresponds to one unit of 1D case (segments between two arrows in Fig. 1(c)). Then shortcuts are established between representative nodes, with quantum operations identical to 1D case. In other words, its definite physical realization relies on which QRP is utilized. Entanglement swapping[30] and purification[31–33] are two most important quantum operations. By entanglement swapping, adjacent entangled links are connected and extended to two representative nodes. To obtain high fidelity entangled states, entanglement purification is required, which extracts nearly perfect entangled states from states with lower degree of entanglement. Alternatively, some QRP use quantum error correction[18–20]. This class of QRP could circumvent the probabilistic fashion of purification-based protocols, and relax the strong requirement of long-lived quantum memory[34–36]. Thus they have potential to extend entanglement to longer distance and speed up communication rate.

Why small-world and scale-free properties are relevant for quantum networks. Despite the great diversity of real-world networks, they share some common features. Most of real-world networks are scale-free networks with small-world property[6,7]. A network is scale-free if the probability to find a node with k links $P(k)$, follows a power law $P(k) \sim k^{-\gamma}$ (for real-world networks, $2 < \gamma < 3$). This is quite different from Poisson distribution for Erdős-Rényi random graphs[6,7,21]. Small-world is an influential concept describing such a phenomenon: despite the large size of networks, on average, any two nodes are separated by relatively short distance, which typically scales as: $\bar{\ell} \sim \ln N$[6,7,22]. These features play a dominant role on dynamic functions of complex networks. One naturally wonders whether the two fundamental characteristics make a difference to quantum networks.

The realistic significance of small-world can be illustrated from the perspective of limited-path-length entanglement percolation[11]. Without quantum processing such as entanglement purification, the fidelity of communication along a long noisy path of imperfect entangled states decreases exponentially[11,16]. Thus it yields a very short distance ℓ of faithful communication, which severely limits effective size of quantum networks N_e. Things seem to be bad. However, if $\bar{\ell} < \ell$, it turns out that most of nodes are reachable within the distance of reliable communication. For d-dimensional regular lattice, $\bar{\ell} \sim N^{1/d}$, and we have $N_e \sim \ell^d$. In contrast, $N_e \sim e^{\ell}$ for small-world networks. We therefore need a quantum small-world network, so that without further quantum processing, this desirable

topological effect alone suffices to effectively mitigate the limitation of noise, and enhance the scalability of quantum networks.

The striking effect of small-world originates from the existence of shortcuts between remote nodes. A quantum network without shortcuts is not small-world network. Consequently, it's rather difficult to extend the influential concept of small-world to large-scale quantum networks, where shortcuts are not directly available. CR can generate a set of shortcuts, which eventually leads to small-world transition at a relatively small nesting level. We will clarify it in subsequent section. While CR can be applied to fractal networks such as Erdős-Rényi random graphs at criticality[26], apparently, such topology is not a realistic option. We propose that QRN are scale-free and fractal observed at arbitrary spacial scale. We justify this proposal with twofold reasons.

Entanglement percolation is a combination of entanglement swapping and classical entanglement percolation[9]. In their framework, the threshold actually depends on the final topology. Notably, the percolation threshold for scale-free networks can be vanishing[37–39]. It means that, as far as scale-free networks, classical entanglement percolation is such a good strategy that the critical amount of entanglement required for the presence of giant cluster is zero in asymptotic limit. Therefore, scale-free QRN has exceptional ability to preserve connectivity in the presence of noise. In addition, it's strongly supported by three facts. Other than local operations, classical communication is indispensable between quantum nodes. And entangled photon pairs can be transmitted through commercial telecom fiber. Thus, to some extent, QRN is embedded in classical communication networks. Whereas both phone call networks and Internet are scale-free networks[6,7]. If QRN is scale-free, we can make full use of the existing network infrastructures, without significantly altering them. Above facts suggest that scale-free network is a plausible and eligible candidate for QRN.

Without loss of generality, we apply CR to a scale-free fractal network generated by the minimal model (see Methods). Let G_n be the nth level CGN. According to renormalization group theory, $G_n = R_{\ell_c}(G_{n-1}) = R_{\ell_c}^n(G_0) = R_{\ell_c^n}(G_0)$. So G_n can be seen as larger-scale network enlarged from G_{n-1}, or equivalently it arises from single level CR with transforming length-scale ℓ_c^n. Larger-scale CGN here collectively act as shortcuts of the underlying smaller-scale ones, which drastically change the topology in such a way that nodes are globally separated by short path of entangled links. This can be further unveiled by the squeezed distance distribution which follows Gaussian distribution (see Fig. 2(b)). Hence, in the end, a hierarchically nested quantum small-world network is produced.

Proof of fractal to small-world transition. We proceed to make it clear whether single or iterative CR will lead to fractal to small-world transition. Two analytical proofs with numerical simulations are provided. A rigorous and reliable method is to observe the behavior of average degree under renormalization flow (see Methods)[40]. In regard to single level CR, the expected transition does not arise. However, it's safe to say that iterative CR can give rise to fractal to small-world transition. Evidences for the transition displayed in Fig. 3 conform above conclusion. Nonetheless, the average diameter of CN, namely, the signature of small-world is unclear.

We begin with analyzing the impact of single level CR, and then generalize it to iterative case. Plugging Eq. (3) into Eq. (1), we immediately obtain the diameter of CGN $D_B(\ell_c) \sim D_0/\ell_c$. In order to compute $D_C(\ell_c)$, the diameter of CN, we suggest a hierarchical routing method which exploits the hierarchical structure and convert it into a routing problem. A path connecting two remote nodes is divided into two parts: one part links one of the nodes and its corresponding hub within the box, and the other part, consisting of only representative nodes, connects the two hubs. The total length of the first part is approximately $\ell_c - 1$, and that of the second is $D_B(\ell_c)$. We thus have
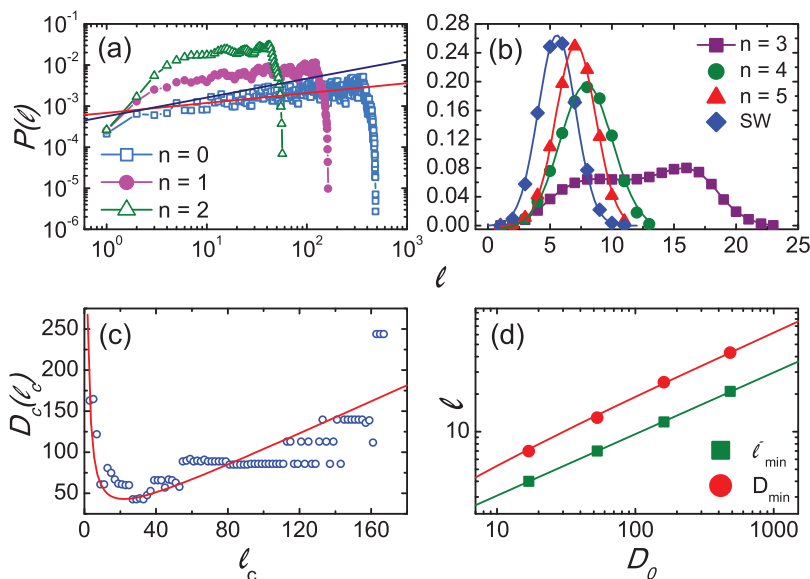
Figure 2 | Statistical properties of CN. In this example, $t = 5$, $N = 9375$, $\ell_c = 3$. (a) and (b) Distance distribution of CN with different nesting levels. (a) is log-log plot of $P(\ell)$ versus ℓ, the slope of the upper line is $d_B - 1 \approx 0.46$ (analytical estimate), and the lower line 0.24 (fitting). (c) Prediction of diameter of single level CN as a function of ℓ_c. (d) Log-log plot of the minimal and average diameter of different size of single level CN.

$$D_C(\ell_c) \approx \frac{D_0}{\ell_c} + \ell_c - 1, \tag{4}$$

see Fig. 2(c). Take note that there is an optimal transforming length-scale, $\ell_o = \sqrt{D_0}$, which yields minimal diameter $D_{min}(N) = 2\sqrt{D_0} - 1$. Meanwhile, numerical simulation gives the corresponding minimal average diameter $\bar{\ell}_{min}(N) = D_{min}(N)/2 \approx \sqrt{D_0}$ (see Fig. 2(d)). An analytical approximation shows that $\bar{\ell}_{min}(N)$ scales as a power law too (see supplemental information). The behavior of $\bar{\ell}_{min}$ suggests that single level CR is unable to trigger the transition, which is consistent with above conclusion.
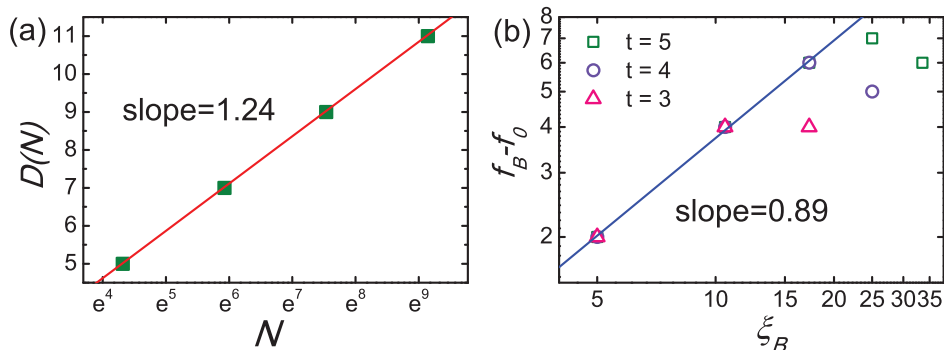
Now let's consider multi-level CR with fixed box size. In analogy with above results, it's easy to obtain the diameter of iterative CN $D(n, \ell_c)$ for small ℓ_c, using Eq. (4) with recursive derivation, we find

$$D(n, \ell_c) \approx \frac{D_0}{\ell_c^n} + n(\ell_c - 1), \tag{5}$$

where n is nesting level of CR. Taking into account finite size effect, Eq. (5) holds on condition that $\ell_c \ll \ell_o$.

Remarkably, the first term decays exponentially, whereas the second increases linearly. Hence, $D(n, \ell_c)$ is governed by the linear term and grows slowly. We readily obtain the criterion for the transition:

$$\ell_c^{n_c} \sim D_0, \tag{6}$$

implying that $D(n_c, \ell_c) \approx n_c(\ell_c - 1)$ for large N. It's desirable that both n_c and $D(n_c, \ell_c)$ increases logarithmically with size of network, since

$$D(n_c, \ell_c) \approx \frac{\ell_c - 1}{d_B \ln \ell_c} \ln N, \tag{7}$$

and

$$n_c \sim \frac{\ln N}{d_B \ln \ell_c}. \tag{8}$$

How to appreciate the implication of n_c now is evident. We identify n_c as critical nesting level, at which small-world is achieved. Direct evidence is shown by Eq. (7). What's more, it's exactly in agreement with simulation result, see Fig. 3(a). Here n_c is inversely proportional to fractal dimension d_B, indicating its topological interdependence. When $d_B = 1$, Eq. (8) reproduces the result of 1D case. While $d_B \to \infty$, $n_c \to 0$, means that the initial network is already small-world, perfectly consistent with conclusion that when $d_B \to \infty$, these networks are small-world without fractality[25,27].

Figure 3 | Evidences for QRN with small-world property. (a) Log-log plot of Eq. (7) with parameters $\ell_c = 3$, and $d_B \approx 1.46$. The analytical prediction (straight line) matches well with numerical simulation (square). (b) Log-log plot of $f_B - f_0$ versus ξ_B.

Discussion

The highlights of our scenario are as follows. Our scenario is a fairly general framework. CR is compatible with various QRP based on the aforementioned principle. In principle, CR is applicable to arbitrary quantum networks with fractal structure, not restricted to the example of scale-free networks. And the transition will arise as long as Eq. (6) or Eq. (8) is satisfied. It's not difficult to check that the unique requirement of the whole derivations is the fractality of QRN. Furthermore, the collective distribution of shortcuts across entire network is mapped to renormalization transformation, where the self-similar fashion of operations is preserved at network level, while the scale-free fractal structure is kept at all length-scales. Each level transformation enlarges underling network into larger-scale QRN. The simultaneous logarithmic scaling of critical nesting level and corresponding diameter suggests that to achieve small-world, CR is operable even for QRN of large size. Moreover, thanks to the scale-free nature, CN is particularly resilient to random failure of quantum nodes.

To summarize, we have generalized one-dimensional quantum repeaters to high fractal dimension by introducing an approach called CR, which relates entanglement distribution over arbitrary fractal QRN to recursive renormalization transformations. We assume that QRN is fractal and scale-free, which is the case for a large number of real-world networks. In spite of the large size of QRN, there exists a relatively small critical nesting level for CR, at which small-world is obtained. Small-world seems to be a necessary element for a scalable quantum network. Our study suggests that concepts and tools from statistical physics will play an important role in the joint study. It has conceived another significant direction which may open new avenue to address the outstanding issue of complexity. That is, quantum simulation of dynamic process on complex networks or design of quantum algorithms for sophisticated questions in network science[41–43]. All of these attempts may dramatically alter the landscape of both fields.

Methods

Minimal model for scale-free networks. The growth of the network is actually the inverse procedure of renormalization[25]. we begin from a triangle at time step $t = 0$.

(i)   At time step $t + 1$, m new nodes are connected to the endpoints of each link l generated at time step t.
(ii) Then, with probability $1 - e$, we remove link l of time step t, and add one new link connecting a new pair of nodes attached to the endpoints of link l.
(iii) Repeat (i) and (ii) recursively until the wanted time step.

This model produces a scale-free network with degree distribution exponent $\gamma = 1 + \ln(2m + 1)/\ln(m + e)$. We are particularly interested in two distinct types of networks with $e = 0$ or $e = 1$, where a pure fractal network with fractal dimension $d_B = \ln(2m + 1)/\ln 3$, and a nonfractal, small-world network (SW) are achieved respectively. For simplicity and without loss of university, here let $m = 2$, $e = 0$, then $d_B \approx 1.46$.

Renormalization group method for proof of fractal to small-world transition.
Literature[40] studied networks constructed by randomly adding links to a fractal network with probability $p(\ell) \sim \ell^{-z}$. Let f_0, f' and f_B be the average degree of the initial, the new and renormalized network respectively. Then

$$f_B - f_0 = (f' - f_0)\xi_B^\lambda, \qquad (9)$$

where $\xi_B = \ell_B^{d_B}$. Let $s = \alpha/d_B$. If $s > 1$, $\lambda = 2 - s$, otherwise $\lambda = 1$. Notice that when $s = 2$, $\lambda = 0$, a stable phase corresponding to fractal network is separated from the unstable phase moving toward complete graph, where small-world is achieved.

With purpose of finding the location of CN in the phase diagram, we have to calculate the exponent λ. In our model, on one hand,

$$p(\ell) \sim \ell^{-d_B}/\ell^{d_B - 1} = \ell^{-(2d_B - 1)}, \qquad (10)$$

so $\alpha \approx 2d_B - 1$, and $s = 2 - 1/d_B$, $\lambda = 2 - s = 1/d_B \approx 0.68$. On the other hand, numerical simulation shows that $\lambda \approx 0.89$ (see Fig. 3(b)). The apparent deviation is mainly caused by the rough approximation that $p(\ell)$ follows a power law (see Fig. 2(a)). In spite of the deviation, it's definite that $\lambda \gg 0$. Thus, CN belongs to the unstable phase and multi-level CR can give rise to fractal to small-world transition.

For single level CR, we have

$$f_B - f_0 = 2\ell_c^{-d_B}N/N_B = 2(\ell_B/\ell_c)^{d_B}. \qquad (11)$$

When $\ell_B > \ell_c$, the links we added will not emerge in the renormalized network with length-scale ℓ_B[40]. We only need to consider one case $\ell_c = \ell_o$, which diverges in the large size limit. So $f_B - f_0 \to 0$, and $\lambda \ll 0$. Thus, coinciding with prediction of equation $\bar{\ell}_{min}(N) = \sqrt{D_0} = N^{1/d_B}$, single level CN stays in the stable phase, the expected transition does not occur.

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Author contributions

Z.-W.W. conceived the research, performed analytical and numerical calculations, and wrote the manuscript. X.-P.H. discussed the manuscript and prepared the figures. B.-H.W. supervised the research.

Additional information