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Estimating neutrosophic finite median employing robust measures of the auxiliary variable

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Our study explores neutrosophic statistics, an extension of classical and fuzzy statistics, to address the challenges of data uncertainty. By leveraging accurate measurements of an auxiliary variable, we can derive precise estimates for the unknown population median. The estimators introduced in this research are particularly useful for analysing unclear, vague data or within the neutrosophic realm. Unlike traditional methods that yield single-valued outcomes, our estimators produce ranges, suggesting where the population parameter is likely to be. We present the suggested generalised estimator's bias and mean square error within a first-order approximation framework. The practicality and efficiency of these proposed neutrosophic estimators are demonstrated through real-world data applications and the simulated data set.

Keywords Neutrosophic statistics, Auxiliary variable, Population median, Robust measures, Mean squared error

Significant strides have been made in recent years in estimating the finite population mean, proportion, and variance using auxiliary information. However, the median, a more robust measure than the mean in the face of exceptionally low or high values, has received attention. Our research stands out by proposing a practical solution for estimating the median of a study variable for a finite population, even in the presence of neutrosophic forms of research and supporting variables with extreme values or outliers.

Several studies have made significant contributions to estimating a finite population's median. Gross¹ used the sample median estimator in various sampling methods. Kuk and Mak² proposed using the known median of the auxiliary variable to estimate the median of the study variable. Francisco and Fuller³ approximated the distribution function of a finite population using the median. Smarandache⁴ advocated using Neutrosophic statistics in uncertain systems. Singh et al.⁵ suggested a generalized family of median estimators in double sampling. These studies have laid the foundation for our research, which aims to build upon these findings and propose a practical solution for estimating the median of a study variable for a finite population.

A few chain ratio-type estimators were introduced by⁶ using the additional knowledge of the range of the auxiliary variable, whereas⁷ discussed the transformed ratio-type estimator using 's' idea. Shokri^{9, 10, 58} presented new approaches to solving second-order initial value problems, providing effectiveness in addressing computational challenges. A generalized median estimator utilizing the transformed auxiliary variable was addressed by¹¹. Lamichhane et al.¹² suggested a unique estimation for the finite population mean using the auxiliary variable's median.

Smarandache¹³ suggested that the sample size may not be accurately specified in neutrosophic statistics and hence presented the neutrosophic logic may fall within the interval $[a, b]$ (unidentified exactly). Sahin¹⁴ and Şahin¹⁵ proposed a new similarity measure based on falsity value between single-valued neutrosophic sets based on the centroid points of transformed single-valued neutrosophic numbers in decision-making. Shokri¹⁶ and Uluçay¹⁷ proposed the similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision-making. Jan¹⁸ proposed multi-criteria decision-making for cubic linguistic information. Aslam^{19, 20} explained the Neutrosophic analysis of variance on neutrosophic data. Using two supplementary variables,²¹ and²² presented difference-type median estimators for obtaining the population median. Chakraborty²³ developed and categorized a trapezoidal bipolar neutrosophic number in decision-making.

For instance,²⁴'s development of complex neutrosophic fuzzy sets contributed to advancing the field of fuzzy sets. In addition, they offered an extensive flowchart of fuzzy sets with extensions, a description of their

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properties, and an explanation of how interval-valued neutrosophic sets function. Haque²⁵ proposed a multi-criteria group decision-making strategy for the cylindrical neutrosophic domain.

Data visualization, analysis, and inference have long used classical statistics (CS). The CS investigates the assumption of data certainty. When the observations' measurements are precise, new methods for handling uncertain data are needed. Fuzzy logic works with data where the variable being researched lacks accurate measurements. They are quickly evolving and often used in settings where decisions are made. Fuzzy statistics examine data with ambiguous, opaque, or uncertain observations but neglect indeterminacy measurement. In this situation, showing a range of specific observations might be possible. The data included inside the indeterminacy interval cannot thus be analyzed using the CS. Neutrosophic statistical techniques are used to analyze the ambiguous neutrosophic data. The neutrosophic logic is used to interpret vague or unclear observations and allows for the measurement of indeterminacy and the determinate part of the observations.

Aslam^{26, 27} elaborated on Neutrosophic Interval Statistics (NIS), Neutrosophic Applied Statistics (NAS), and Neutrosophic Statistical Quality control (NSQC), respectively. Uluçay²⁸ suggested the idea of interval-valued refined neutrosophic sets and their applications. A large number of neutrosophic sets are described in the literature.

Tahir²⁹ addressed a sampling gap by conducting a study to estimate a population's characteristics in a neutrosophic environment. They presented estimators of the neutrosophic ratio-type for estimating the finite population mean utilizing the additional information. Uluçay^{30, 28} presented Q-neutrosophic soft graphs in operations management and communication networks. Vishwakarma and Singh³¹ proposed a neutrosophic ranked set sampling strategy for estimating the population mean under uncertainty using neutrosophic auxiliary information.

Table 1 illustrates the versatility and utility of neutrosophic logic in tackling problems across various disciplines, especially in situations where traditional binary logic falls short due to vagueness, uncertainty, or contradictory information.

Despite an extensive review of existing research, a study still needs to address the challenge of estimating the unknown population median in survey sampling when additional variables are introduced under neutrosophic information. The effectiveness of estimators remains to be determined in scenarios where the study variable and supporting variables take on neutrosophic forms, and the dataset includes extreme values or outliers. The lack of a practical solution for median estimation in such cases underscores the novelty and importance of our proposed neutrosophic median estimation method based on reliable measures of the auxiliary variable that are already known.

Neutrosophic statistics are applied in decision-making, risk assessment, uncertainty modelling, image processing, medical diagnosis, finance, and engineering for robust analysis as discussed in Table 1.

The paper is organized in a way that Section "The neutrosophic statistics" elaborates the details of Neutrosophic statistics along with symbols and notations. The adapted and proposed Neutrosophic median estimators along with the efficiency comparison are given in Section "The neutrosophic median estimators under simple random sampling". The numerical and graphical results related to real-life and simulated data sets are presented in Section "Real-life application". Finally, the proposed work is concluded in Section "Conclusion".

The neutrosophic statistics

Neutrosophic statistics, a unique approach, are designed to handle datasets with a degree of ambiguity or partial information. This method allows for conflicting beliefs and accommodates a range of uncertain numbers that may represent some observations, including an exact measurement. In contrast, traditional statistics falter when faced with uncertainty. This is where the intriguing potential of neutrosophic statistics comes into play, offering a fresh perspective on data analysis.

In real-world problems, the population parameters are often unknown. In such cases, statistical inference methods may need to be more practical. Instead, acceptable estimates are used, resolving the issue of an unknown parameter value by estimating its values. This pragmatic approach reassures the statistician that the derived data are vague but still useful. Neutrosophic statistics, with their ability to calculate the best interval value with the minimum mean square error, offer a reliable solution to these problems.

Domain	Problem solved	Description
Medical diagnosis	Disease identification	Improving the accuracy of diagnosing diseases by handling uncertain or incomplete medical data
Image processing	Image segmentation and enhancement	Enhancing image quality and segmentation in environments with vague and indistinct boundaries
Decision making	Multi-criteria decision making (MCDM)	Facilitating decision-making processes in scenarios with incomplete, inconsistent, or uncertain information
Engineering	Fault diagnosis and system reliability analysis	Assessing system reliability and diagnosing faults in complex engineering systems under uncertainty
Environmental science	Environmental impact assessment	Evaluating environmental impacts with imprecise data, aiding in more effective decision-making
Social sciences	Opinion mining and sentiment analysis	Analyzing sentiments and opinions in social media data, where opinions can be indeterminate or inconsistent
Robotics	Robot localization and navigation	Improving robot navigation in uncertain environments by dealing with imprecise sensor data
Economics	Economic forecasting	Enhancing economic forecasting models by incorporating uncertainty in economic data

Table 1. Some real-life problems and domains under neutrosophic logic.

Previous study efforts provided a limited range of neutrosophic observations, including quantifiable neutrosophic data. Furthermore^{32–40} discussed numerous approaches, such as interval-based approaches, Triangular or trapezoidal fuzzy numbers, and single-valued fuzzy numbers, exist to express the range of neutrosophic numbers along with Optimal trajectories in reproduction models of economic dynamics. He⁴¹ proposed a fractal model for internal temperature response in porous concrete, advancing understanding in applied mathematics. Iskandarov and Komartsova^{42,57} investigated integral perturbations' influence on boundedness in fourth-order linear differential equations. Khankishiyev⁴³ employed finite differences to solve loaded differential equations, while^{44, 56} explored dark energy solutions without a cosmological constant. Furthermore,⁴⁵ established conditions for complete monotonicity in the differential functions involving trigamma functions.

Let the neutrosophic range is $T_N = T_L + T_U \ell_N$ with $\ell_N \in [\ell_L, \ell_U]$, the neutrosophic variable T_N indicates the neutrosophic samples selected from a population having imprecise, ambiguous and unclear measurements. Thus, for the neutrosophic data in the interval form, we use notation $T_N \in [a, b]$, where a and b are the lower and upper values of the neutrosophic data, respectively.

Figure 1 depicts the approach to applying the proposed estimation methods in neutrosophic statistics. This workflow developed a few neutrosophic estimators to estimate the finite population median in the presence of supplementary data, which are well suited for overcoming the sample indeterminacy problem.

Symbols and notations

Suppose a neutrosophic random sample of size $n_N \in [n_L, n_U]$, selected from a finite population $U = \{U_1, U_2, \dots, U_N\}$ of N recognisable units. Assume $y_N(i)$ and $x_N(i), i = (1, 2, \dots, n)$ represent the i^{th} sampled unit's values of the neutrosophic data of the study variable $y_{N(i)} \in [y_L, y_U]$ and the supplementary variable $x_{N(i)} \in [x_L, x_U]$, respectively. Let M_{y_N} be the neutrosophic variable of interest and \hat{M}_{y_N} and \hat{M}_{x_N} represent the sample median that corresponds to the population median M_{y_N} and M_{x_N} respectively with the probability density functions $f_{y_N}(M_{y_N})$ and $f_{x_N}(M_{x_N})$, respectively, where $\hat{M}_{y_N} \in [\hat{M}_{y_L}, \hat{M}_{y_U}]$ and $\hat{M}_{x_N} \in [\hat{M}_{x_L}, \hat{M}_{x_U}]$. Let $\rho_{yxN} = \rho(\hat{M}_{y_N}, \hat{M}_{x_N}) = 4\rho_{11}(y_N, x_N) - 1$ represent the neutrosophic population correlation coefficient between the neutrosophic sample medians ranging from -1 to $+1$ as ρ_{11} increases from 0 to 0.5, where $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$

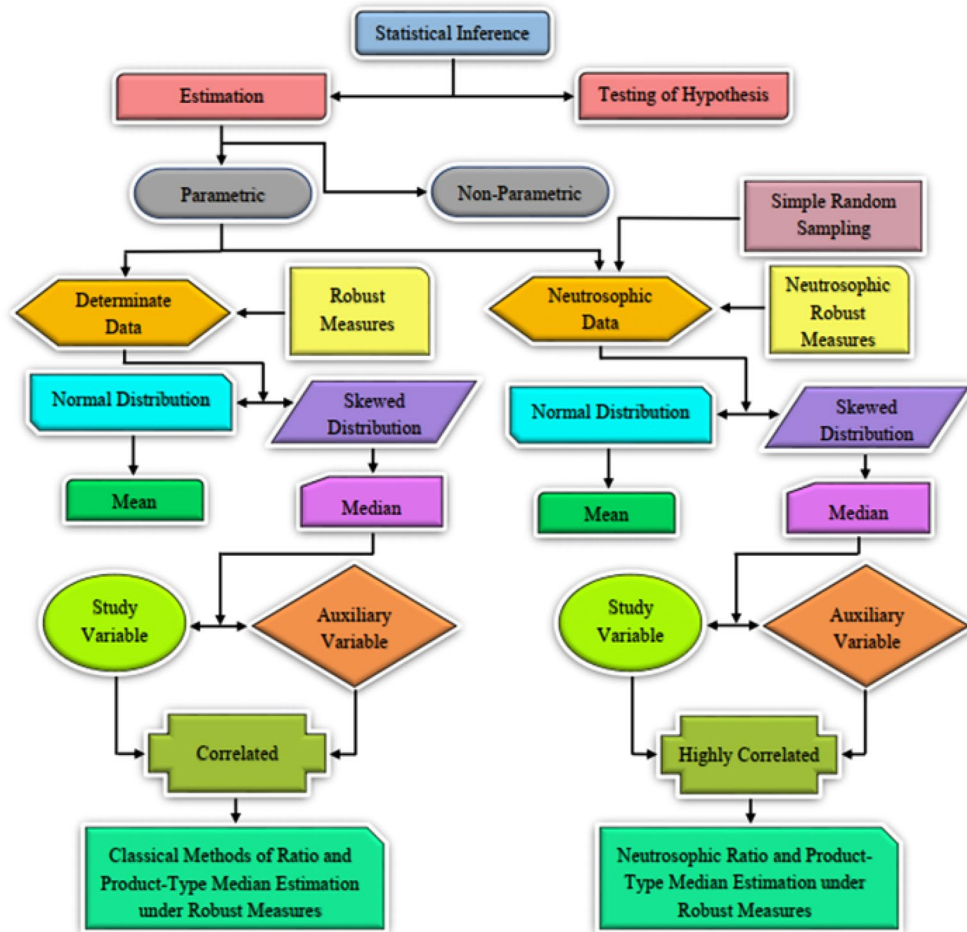


Figure 1. Workflow of the parameter estimation.

such that $P_{11}(y_N, x_N) = P(y_N \leq M_{yN} \cap x_N \leq M_{xN})$. Similarly, $Bias(\hat{M}_{yN}) \in [Bias_L, Bias_U]$ and $MSE(\hat{M}_{yN}) \in [MSE_L, MSE_U]$ belong to the neutrosophic sets.

To get the characteristics of estimators, the relative error terms are defined as follows: Suppose $e_{0N} = (\hat{M}_{yN} - M_{yN})/M_{yN}$, $e_{1N} = (\hat{M}_{xN} - M_{xN})/M_{xN}$ are the neutrosophic errors where $e_{0N} \in [e_{0L}, e_{0U}]$ and $e_{1N} \in [e_{1L}, e_{1U}]$ such that $E(e_{iN}) = 0$ ($i = 0, 1$), $E(e_{0N}^2) = \lambda_N C_{M_{yN}}^2$, $E(e_{1N}^2) = \lambda_N C_{M_{xN}}^2$, $E(e_{0N}e_{1N}) = \lambda_N C_{M_{yxN}}$, where $C_{M_{yxN}} = \rho_{yxN} C_{M_{yN}} C_{M_{xN}}$, $C_{M_{yN}} = \{M_{yN} f_{yN}(M_{yN})\}^{-1}$, $C_{M_{xN}} = \{M_{xN} f_{xN}(M_{xN})\}^{-1}$ are the neutrosophic coefficients of variation, respectively. Let $C_{M_{yxN}} \in [C_{M_{yxL}}, C_{M_{yxU}}]$, $C_{M_{yN}} \in [C_{M_{yL}}, C_{M_{yU}}]$, $C_{M_{xN}} \in [C_{M_{xL}}, C_{M_{xU}}]$ and $\lambda_N = \frac{1}{4} \left(\frac{1}{n_N} - \frac{1}{N_N} \right)$, where $\lambda_N \in [\lambda_L, \lambda_U]$.

The neutrosophic median estimators under simple random sampling

First, we present a few adapted neutrosophic median estimators using auxiliary information under simple random sampling to address uncertainty and neutrosophic data.

Adapted median estimators with auxiliary variable

- (i) Motivated by¹, we propose a neutrosophic traditional median estimator and its variance, along with the expression of variance is given by

$$\hat{M}_{0N} = \hat{M}_{yN} \tag{1}$$

$$Var(\hat{M}_{0N}) = \lambda_N M_{yN}^2 C_{M_{yN}}^2 \tag{2}$$

- (ii) Motivated by², we developed a novel neutrosophic traditional ratio estimator, along with the expressions of Bias and MSE are

$$\hat{M}_{RN} = \hat{M}_{yN} \left(\frac{M_{xN}}{\hat{M}_{xN}} \right) \tag{3}$$

$$Bias(\hat{M}_{RN}) \cong \lambda_N M_{yN} \{ C_{M_{xN}}^2 - C_{M_{yxN}} \} \tag{4}$$

and

$$MSE(\hat{M}_{RN}) \cong \lambda_N M_{yN}^2 \{ C_{M_{yN}}^2 + C_{M_{xN}}^2 - 2C_{M_{yxN}} \} \tag{5}$$

The ratio estimator (\hat{M}_{RN}) performs better than \hat{M}_{0N} if $\rho_{yxN} > 0.5 \frac{C_{M_{xN}}}{C_{M_{yN}}}$.

- (iii) Motivated by⁴⁶, the neutrosophic exponential ratio-type estimator, along with the expressions of Bias and MSE are given by

$$\hat{M}_{EN} = \hat{M}_{yN} \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) \tag{6}$$

$$Bias(\hat{M}_{EN}) \cong M_{yN} \lambda_N \left(\frac{3}{8} C_{M_{xN}}^2 - \frac{1}{2} C_{M_{yxN}} \right) \tag{7}$$

and

$$MSE(\hat{M}_{EN}) \cong M_{yN}^2 \lambda_N \left(C_{M_{yN}}^2 + \frac{1}{4} C_{M_{xN}}^2 - C_{M_{yxN}} \right) \tag{8}$$

The exponential ratio estimator (\hat{M}_{EN}) is more efficient than \hat{M}_{0N} and \hat{M}_{RN} if $\rho_{yxN} > 0.25 \frac{C_{M_{xN}}}{C_{M_{yN}}}$ and $\rho_{yxN} < 0.75 \frac{C_{M_{xN}}}{C_{M_{yN}}}$, respectively.

- (iv) The adapted neutrosophic difference estimator along with the expression of variance is given by

$$\hat{M}_{D0N} = \hat{M}_{yN} + d_{0N} (M_{xN} - \hat{M}_{xN}) \tag{9}$$

At the optimal value of d_{0N} , which is $d_{0N(opt)} = \frac{M_{yN} \rho_{yxN} C_{M_{yN}}}{M_{xN} C_{M_{xN}}}$, the minimum MSE of \hat{M}_{D0N} , is given by

$$Var(\hat{M}_{D0N})_{min} \cong M_{yN}^2 C_{M_{yN}}^2 \lambda_N (1 - \rho_{yxN}^2) \tag{10}$$

(V) Adapted from²², difference-type estimators, along with the expressions of Bias and minimum mean square errors, are given by

$$\hat{M}_{D_{1N}} = d_{1N}\hat{M}_{yN} + d_{2N}(M_{xN} - \hat{M}_{xN}) \tag{11}$$

$$\hat{M}_{D_{2N}} = \left\{ d_{3N}\hat{M}_{yN} + d_{4N}(M_{xN} - \hat{M}_{xN}) \right\} \left(\frac{M_{xN}}{\hat{M}_{xN}} \right), \tag{12}$$

$$\hat{M}_{D_{3N}} = \left\{ d_{5N}\hat{M}_{yN} + d_{6N}(M_{xN} - \hat{M}_{xN}) \right\} \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right), \tag{13}$$

and

$$\hat{M}_{D_{4N}} = \left\{ d_{7N}\hat{M}_{yN} + d_{8N}(M_{xN} - \hat{M}_{xN}) \right\} \exp \left(\frac{M_{xN}}{\hat{M}_{xN}} - 1 \right), \tag{14}$$

where d_{iN} ($i = 1 - 8$) are constants determined below by optimality considerations as

$$\begin{aligned} d_{1N(opt)} &= \frac{B_{0N}}{A_{0N}B_{0N} - C_{0N}^2 + B_{0N}}, d_{2N(opt)} = \frac{M_{yN}}{M_{xN}} \frac{C_{0N}}{A_{0N}B_{0N} - C_{0N}^2 + B_{0N}}, \\ d_{3N(opt)} &= \frac{B_{1N}(C_{1N} - D_{1N} + 1)}{A_{1N}B_{1N} - D_{1N}^2 + B_{1N}}, d_{4N(opt)} = \frac{M_{yN}}{M_{xN}} \frac{(A_{1N}B_{1N} - C_{1N}D_{1N} + B_{1N} - D_{1N})}{(A_{1N}B_{1N} - D_{1N}^2 + B_{1N})}, \\ d_{5N(opt)} &= \frac{(B_{2N}C_{2N} - D_{2N}E_{2N} + B_{2N})}{(A_{2N}B_{2N} - E_{2N}^2 + B_{2N})}, d_{6N(opt)} = \frac{M_{yN}}{M_{xN}} \frac{(A_{2N}D_{2N} - C_{2N}E_{2N} + D_{2N} - E_{2N})}{(A_{2N}B_{2N} - E_{2N}^2 + B_{2N})}, \\ d_{7N(opt)} &= \frac{B_{3N}(C_{3N} - D_{3N} + 1)}{A_{3N}B_{3N} - D_{3N}^2 + B_{3N}}, d_{8N(opt)} = \frac{M_{yN}}{M_{xN}} \frac{(A_{3N}B_{3N} - C_{3N}D_{3N} + B_{3N} - D_{3N})}{(A_{3N}B_{3N} - D_{3N}^2 + B_{3N})}, \text{ where } A_{0N} = \lambda_N C_{M_{yN}}^2, B_{0N} = \\ &\lambda_N C_{M_{xN}}^2, C_{0N} = \lambda_N C_{M_{yxN}}, A_{1N} = \lambda_N (C_{M_{yN}}^2 + 3C_{M_{xN}}^2 - 4C_{M_{yxN}}), B_{1N} = \lambda_N C_{M_{xN}}^2, C_{1N} = \lambda_N (C_{M_{xN}}^2 - C_{M_{yxN}}), \\ &D_{1N} = \lambda_N (2C_{M_{xN}}^2 - C_{M_{yxN}}), A_{2N} = \lambda_N (C_{M_{yN}}^2 + C_{M_{xN}}^2 - 2C_{M_{yxN}}), B_{2N} = \lambda_N C_{M_{xN}}^2, C_{2N} = \lambda_N \left(\frac{3}{8} C_{M_{xN}}^2 \right. \\ &\left. - \frac{1}{2} C_{M_{yxN}} \right), D_{2N} = \lambda_N C_{M_{xN}}^2 / 2, E_{2N} = \lambda_N (C_{M_{xN}}^2 - C_{M_{yxN}}), A_{3N} = \lambda_N (C_{M_{yN}}^2 + 4C_{M_{xN}}^2 - 4C_{M_{yxN}}), B_{3N} = \\ &\lambda_N C_{M_{xN}}^2, C_{3N} = \lambda_N \left(\frac{3}{2} C_{M_{xN}}^2 - C_{M_{yxN}} \right) \text{ and } D_{3N} = \lambda_N (2C_{M_{xN}}^2 - C_{M_{yxN}}). \end{aligned}$$

$$\text{Bias}(\hat{M}_{D_{1N}}) \cong (d_{1N} - 1)M_{yN}, \tag{15}$$

$$\text{Bias}(\hat{M}_{D_{2N}}) \cong (d_{3N} - 1)M_{yN} + d_{3N}M_{yN}C_{1N} + d_{4N}M_{xN}B_{1N}, \tag{16}$$

$$\text{Bias}(\hat{M}_{D_{3N}}) \cong (d_{5N} - 1)M_{yN} + d_{5N}M_{yN}C_{2N} + d_{6N}M_{xN}D_{2N}, \tag{17}$$

$$\text{Bias}(\hat{M}_{D_{4N}}) \cong (d_{7N} - 1)M_{yN} + d_{7N}M_{yN}C_{3N} + d_{8N}M_{xN}B_{3N}, \tag{18}$$

$$\text{MSE}(\hat{M}_{D_{1N}})_{\min} \cong M_{yN}^2 \left\{ 1 - \frac{B_{0N}}{A_{0N}B_{0N} - C_{0N}^2 + B_{0N}} \right\}, \tag{19}$$

$$\text{MSE}(\hat{M}_{D_{2N}})_{\min} \cong M_{yN}^2 \left\{ 1 - \frac{A_{1N}B_{1N}^2 + B_{1N}C_{1N}^2 - 2B_{1N}C_{1N}D_{1N} + B_{1N}^2 + 2B_{1N}C_{1N} - 2B_{1N}D_{1N} + B_{1N}}{A_{1N}B_{1N} - D_{1N}^2 + B_{1N}} \right\}, \tag{20}$$

$$\text{MSE}(\hat{M}_{D_{3N}})_{\min} \cong M_{yN}^2 \left\{ 1 - \frac{A_{2N}D_{2N}^2 + B_{2N}C_{2N}^2 - 2C_{2N}D_{2N}E_{2N} + 2B_{2N}C_{2N} + D_{2N}^2 - 2D_{2N}E_{2N} + B_{2N}}{A_{2N}B_{2N} - E_{2N}^2 + B_{2N}} \right\}, \tag{21}$$

and

$$\text{MSE}(\hat{M}_{D_{4N}})_{\min} \cong M_{yN}^2 \left\{ 1 - \frac{A_{3N}B_{3N}^2 + B_{3N}C_{3N}^2 - 2B_{3N}C_{3N}D_{3N} + B_{3N}^2 + 2B_{3N}C_{3N} - 2B_{3N}D_{3N} + B_{3N}}{A_{3N}B_{3N} - D_{3N}^2 + B_{3N}} \right\}. \tag{22}$$

The proposed generalized neutrosophic median estimator

Traditional estimators, often hindered by their reliance on historical data, struggle with accuracy, particularly with outliers. This section introduces advanced neutrosophic estimators for accurately predicting a finite population's median. These estimators blend unique metrics like quartile deviation and interquartile range, enhancing data distribution analysis and outlier exclusion through robust scaling, employing decile means, the Hodges-Lehmann estimator, and tri-mean for reliable median estimation. The tri-mean proposed by⁴⁷, the Hodges-Lehmann estimator proposed by⁴⁸ and the decile means proposed by⁴⁹ are the three robust metrics we used in this study. For further information about these robust measures, readers can see⁵⁰ and⁵¹ for details.

Motivated by⁵², we develop a neutrosophic generalized ratio-type estimator of finite population median as

$$T_{i(d)N} = \hat{M}_{yN} \left[\left\{ m_{1N} \left(\frac{\psi_N \hat{M}_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right)^{\alpha_3} \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) \right\} + \left\{ m_{2N} \left(\frac{\psi_N M_{xN} + \delta_N}{\psi_N \hat{M}_{xN} + \delta_N} \right)^{\alpha_4} \right\} \right], \tag{23}$$

where $T_{i(d)N} \in [T_{i(d)L}, T_{i(d)U}]$, $m_{1N} \in [m_{1L}, m_{1U}]$ and $m_{2N} \in [m_{2L}, m_{2U}]$ are suitable neutrosophic constants, where α_3 and α_4 take the values 1, -1, 2, -2 for developing new estimators.

Note $\psi_N \in [\psi_L, \psi_U]$ and $\delta_N \in [\delta_L, \delta_U]$ are neutrosophic functions of the known robust and non-conventional measures related to the variable X_N . Robust measures associated with X_N are:

- (i) Tri-mean: $T_{MN} = (Q_{1N} + 2Q_{2N} + Q_{3N})/4, T_{MN} \in [T_{ML}, T_{MU}]$
- (ii) Hodges–Lehman: $H_{LN} = Median((x_{jN} + x_{kN})/2), 1 \leq j \leq k \leq N, H_{LN} \in [H_{LL}, H_{LU}]$
- (iii) Decile mean: $D_{MN} = \sum_{i=1}^9 D_{iN}/9, D_{MN} \in [D_{ML}, D_{MU}]$

The non-conventional measures (i.e., interquartile range, midrange, quartile average and quartile deviation) of the supplementary variable are as follows:

- (iv) Interquartile range: $Q_{RN} = Q_{3N} - Q_{1N}, Q_{RN} \in [Q_{RL}, Q_{RU}]$
- (v) Midrange: $M_{RN} = ((x_{(1)N} + x_{(N)N})/2), M_{RN} \in [M_{RL}, M_{RU}]$
- (vi) Quartile average: $Q_{AN} = (Q_{3N} + Q_{1N})/2, Q_{AN} \in [Q_{AL}, Q_{AU}]$
- (vii) Quartile deviation $Q_{DN} = (Q_{3N} - Q_{1N})/2, Q_{DN} \in [Q_{DL}, Q_{DU}]$

where $Q_{1N} \in [Q_{1L}, Q_{1U}]$, $Q_{2N} \in [Q_{2L}, Q_{2U}]$ and $Q_{3N} \in [Q_{3L}, Q_{3U}]$ are the neutrosophic first, second and third quartiles, respectively and $D_{iN} \in [D_{iL}, D_{iU}]$ is the neutrosophic decile.

By putting different values of α_i (for $i = 3, 4$) into (23), we get the following families of estimators as.

i. At $\alpha_3 = 1$ and $\alpha_4 = 2$, the proposed family of estimators reduces to

$$T_{i(d)N}^{\ominus} = \hat{M}_{yN} \left[\left\{ m_{1N} \left(\frac{\psi_N \hat{M}_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right) \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) \right\} + \left\{ m_{2N} \left(\frac{\psi_N M_{xN} + \delta_N}{\psi_N \hat{M}_{xN} + \delta_N} \right)^2 \right\} \right]. \tag{24}$$

ii. At $\alpha_3 = -1$ and $\alpha_4 = -1$; the proposed family of estimators reduces to

$$T_{i(d)N}^{\oplus} = \hat{M}_{yN} \left[\left\{ m_{1N} \left(\frac{\psi_N M_{xN} + \delta_N}{\psi_N \hat{M}_{xN} + \delta_N} \right) \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) \right\} + \left\{ m_{2N} \left(\frac{\psi_N \hat{M}_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right) \right\} \right]. \tag{25}$$

iii. At $\alpha_3 = -1$ and $\alpha_4 = -2$, the proposed family of estimators becomes

$$T_{i(d)N}^{\otimes} = \hat{M}_{yN} \left[\left\{ m_{1N} \left(\frac{\psi_N M_{xN} + \delta_N}{\psi_N \hat{M}_{xN} + \delta_N} \right) \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) \right\} + \left\{ m_{2N} \left(\frac{\psi_N \hat{M}_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right)^2 \right\} \right]. \tag{26}$$

iv. At $\alpha_3 = 2$ and $\alpha_4 = 2$, the proposed family of estimators reduces to

$$T_{i(d)N}^{\circledast} = \hat{M}_{yN} \left[\left\{ m_{1N} \left(\frac{\psi_N \hat{M}_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) \right\} + \left\{ m_{2N} \left(\frac{\psi_N M_{xN} + \delta_N}{\psi_N \hat{M}_{xN} + \delta_N} \right)^2 \right\} \right]. \tag{27}$$

v. At $\alpha_3 = -2$ and $\alpha_4 = -1$, the proposed family of estimators reduces to

$$T_{i(d)N}^{\circledcirc} = \hat{M}_{yN} \left[\left\{ m_{1N} \left(\frac{\psi_N M_{xN} + \delta_N}{\psi_N \hat{M}_{xN} + \delta_N} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) \right\} + \left\{ m_{2N} \left(\frac{\psi_N \hat{M}_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right) \right\} \right]. \tag{28}$$

When we use robust measures with linear combinations of the median, quartile deviation, midrange, interquartile range, and quartile average of the supplementary variable in (23), we get different series of estimators such as $T_{i(d)N}^{\ominus}, T_{i(d)N}^{\oplus}, T_{i(d)N}^{\otimes}, T_{i(d)N}^{\circledast}$ and $T_{i(d)N}^{\circledcirc}$. Few members of the family of estimators $T_{i(d)N}^{\circledcirc}$ are given in Table 2. Putting the same values of ψ_N and δ_N in $T_{i(d)N}^{\ominus}, T_{i(d)N}^{\oplus}, T_{i(d)N}^{\otimes}, T_{i(d)N}^{\circledast}$ and $T_{i(d)N}^{\circledcirc}$, we obtain several estimators.

We can obtain several optimal estimators by placing suitable constants or known conventional parameters of the supplementary variable in place of ψ_N and δ_N into (23). Conventional parameters related to the supplementary variable X_N are variance, standard deviation, coefficient of variation, coefficient of skewness, coefficient of kurtosis, coefficient of correlation, and so forth.

ψ_N	δ_N	i	$T_{i(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{\psi_N M_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{\psi_N \hat{M}_{xN} + \delta_N}{\psi_N M_{xN} + \delta_N} \right) \right]$
Q_{AN}	T_{MN}	1	$T_{1(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{Q_{AN} M_{xN} + T_{MN}}{Q_{AN} M_{xN} + T_{MN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{Q_{AN} \hat{M}_{xN} + T_{MN}}{Q_{AN} M_{xN} + T_{MN}} \right) \right]$
M_{RN}	T_{MN}	2	$T_{2(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{M_{RN} M_{xN} + T_{MN}}{M_{RN} M_{xN} + T_{MN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{M_{RN} \hat{M}_{xN} + T_{MN}}{M_{RN} M_{xN} + T_{MN}} \right) \right]$
H_{LN}	T_{MN}	3	$T_{3(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{H_{LN} M_{xN} + T_{MN}}{H_{LN} M_{xN} + T_{MN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{H_{LN} \hat{M}_{xN} + T_{MN}}{H_{LN} M_{xN} + T_{MN}} \right) \right]$
T_{MN}	H_{LN}	4	$T_{4(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{T_{MN} M_{xN} + H_{LN}}{T_{MN} M_{xN} + H_{LN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{T_{MN} \hat{M}_{xN} + H_{LN}}{T_{MN} M_{xN} + H_{LN}} \right) \right]$
Q_{AN}	H_{LN}	5	$T_{5(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{Q_{AN} M_{xN} + H_{LN}}{Q_{AN} M_{xN} + H_{LN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{Q_{AN} \hat{M}_{xN} + H_{LN}}{Q_{AN} M_{xN} + H_{LN}} \right) \right]$
H_{LN}	Q_{RN}	6	$T_{6(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{H_{LN} M_{xN} + Q_{RN}}{H_{LN} M_{xN} + Q_{RN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{H_{LN} \hat{M}_{xN} + Q_{RN}}{H_{LN} M_{xN} + Q_{RN}} \right) \right]$
M_{RN}	H_{LN}	7	$T_{7(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{M_{RN} M_{xN} + H_{LN}}{M_{RN} M_{xN} + H_{LN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{M_{RN} \hat{M}_{xN} + H_{LN}}{M_{RN} M_{xN} + H_{LN}} \right) \right]$
H_{LN}	D_{MN}	8	$T_{8(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{H_{LN} M_{xN} + D_{MN}}{H_{LN} M_{xN} + D_{MN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{H_{LN} \hat{M}_{xN} + D_{MN}}{H_{LN} M_{xN} + D_{MN}} \right) \right]$
D_{MN}	T_{MN}	9	$T_{9(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{D_{MN} M_{xN} + T_{MN}}{D_{MN} M_{xN} + T_{MN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{D_{MN} \hat{M}_{xN} + T_{MN}}{D_{MN} M_{xN} + T_{MN}} \right) \right]$
M_{RN}	D_{MN}	10	$T_{10(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{M_{RN} M_{xN} + D_{MN}}{M_{RN} M_{xN} + D_{MN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{M_{RN} \hat{M}_{xN} + D_{MN}}{M_{RN} M_{xN} + D_{MN}} \right) \right]$
Q_{AN}	D_{MN}	11	$T_{11(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{Q_{AN} M_{xN} + D_{MN}}{Q_{AN} M_{xN} + D_{MN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{Q_{AN} \hat{M}_{xN} + D_{MN}}{Q_{AN} M_{xN} + D_{MN}} \right) \right]$
D_{MN}	Q_{AN}	12	$T_{12(d)N}^{\odot} = \hat{M}_{yN} \left[m_{1N} \left(\frac{D_{MN} M_{xN} + Q_{AN}}{D_{MN} M_{xN} + Q_{AN}} \right)^2 \exp \left(\frac{M_{xN} - \hat{M}_{xN}}{M_{xN} + \hat{M}_{xN}} \right) + m_{2N} \left(\frac{D_{MN} \hat{M}_{xN} + Q_{AN}}{D_{MN} M_{xN} + Q_{AN}} \right) \right]$

Table 2. Some members of $T_{i(d)N}^{\odot}$.

Bias, MSE, and minimum MSE of the proposed neutrosophic generalized family of estimators $T_{i(d)N}$ in terms of e_{0N} and e_{1N} are expressed as,

$$Bias(T_{i(d)N}) = M_{yN} \left[m_{1N} \left(1 + \frac{\lambda_N C_{M_{xN}}^2}{2} \left(\frac{3}{4} - \alpha_3 \theta_N + \alpha_3 (\alpha_3 - 1) \theta_N^2 \right) + \lambda_N \rho_N C_{M_{yN}} C_{M_{xN}} \left(\alpha_3 \theta_N - \frac{1}{2} \right) \right) + m_{2N} \left(1 + \frac{\alpha_4 (\alpha_4 + 1) \theta_N^2 \lambda_N C_{M_{xN}}^2}{2} - \alpha_4 \theta_N \lambda_N \rho_N C_{M_{yN}} C_{M_{xN}} \right) - 1 \right] \tag{29}$$

where $\theta_N = \psi_N M_{xN} / (\psi_N M_{xN} + \delta_N)$.

The MSE of suggested estimator up to the first order of approximation as

$$MSE(T_{i(d)N}) \cong M_{yN}^2 [1 + m_{1N}^2 A_{1N} + m_{2N}^2 A_{2N} + 2m_{1N} m_{2N} A_{3N} - 2m_{1N} A_{4N} - 2m_{2N} A_{5N}], \tag{30}$$

$$\text{where } A_{1N} = \left[1 + \lambda_N \left\{ C_{M_{yN}}^2 + C_{M_{xN}}^2 (1 + 2\alpha_3^2 \theta_N^2 - \alpha_3 \theta_N^2 - 2\alpha_3 \theta_N) - 2\rho_N C_{M_{yN}} C_{M_{xN}} (1 - 2\alpha_3 \theta_N) \right\} \right]$$

$$A_{2N} = [1 + \lambda_N \{ C_{M_{yN}}^2 + \theta_N^2 C_{M_{xN}}^2 (2\alpha_4^2 + \alpha_4) - 4\alpha_4 \theta_N \rho_N C_{M_{yN}} C_{M_{xN}} \}],$$

$$A_{3N} = \left[1 + \lambda_N \left\{ C_{M_{yN}}^2 - \rho_N C_{M_{yN}} C_{M_{xN}} (2\theta_N (\alpha_4 - \alpha_3) + 1) - C_{M_{xN}}^2 \left(\alpha_3 \alpha_4 \theta_N^2 - \frac{3}{8} + \frac{\alpha_3 \theta_N}{2} - \frac{\alpha_4 \theta_N}{2} - \frac{\alpha_4 (\alpha_4 + 1) \theta_N^2}{2} - \frac{1}{2} \alpha_3 (\alpha_3 - 1) \theta_N^2 \right) \right\} \right]$$

$$A_{4N} = \left[1 + \lambda_N \rho_N C_{M_{yN}} C_{M_{xN}} \left(\alpha_3 \theta_N - \frac{1}{2} \right) + \lambda_N \frac{C_{M_{xN}}^2}{2} \left(\frac{3}{4} - \alpha_3 \theta_N + \alpha_3 (\alpha_3 - 1) \theta_N^2 \right) \right] \text{ and } A_{5N} = \left[1 + \frac{1}{2} \alpha_4 (\alpha_4 + 1) \theta_N^2 \lambda_N C_{M_{xN}}^2 - \alpha_4 \theta_N \lambda_N \rho_N C_{M_{yN}} C_{M_{xN}} \right].$$

The minimum MSE at the optimum values $m_{1N(opt)} = \frac{(A_{2N} A_{4N} - A_{3N} A_{5N})}{(A_{1N} A_{2N} - A_{3N}^2)}$ and $m_{2N(opt)} = \frac{(A_{1N} A_{5N} - A_{3N} A_{4N})}{(A_{1N} A_{2N} - A_{3N}^2)}$, is given by

$$MSE(T_{i(d)N})_{\min} \cong M_{yN}^2 \left[1 - \frac{(A_{2N}A_{4N}^2 + A_{1N}A_{5N}^2 - 2A_{3N}A_{4N}A_{5N})}{(A_{1N}A_{2N} - A_{3N}^2)} \right] \quad (31)$$

Efficiency comparison

(i) By comparing (2) and (23), $Var(\hat{M}_{0N}) > MSE(T_{i(d)N})_{\min}$ if $[\Theta_{2N}(\lambda_N M_{yN}^2 C_{M_{yN}}^2 - 1) + \Theta_{1N}] > 0$, where $\Theta_{1N} = A_{2N}A_{4N}^2 + A_{1N}A_{5N}^2 - 2A_{3N}A_{4N}A_{5N}$ and $\Theta_{2N} = A_{1N}A_{2N} - A_{3N}^2$.

(ii) By comparing (5) and (23), $MSE(\hat{M}_{RN}) > MSE(T_{i(d)N})_{\min}$ if

$$[\Theta_{2N}(\lambda_{hN} C_{M_{yN}}^2 + \lambda_N C_{M_{xN}}^2 - 2\lambda_N C_{M_{yxN}} - 1) + \Theta_{1N}] > 0$$

(iii) By comparing (8) and (23), $MSE(\hat{M}_{EN}) > MSE(T_{i(d)N})_{\min}$ if

$$[\Theta_{2N}(\lambda_N C_{M_{yN}}^2 + \frac{1}{4}\lambda_N C_{M_{xN}}^2 - \lambda_N C_{M_{yxN}} - 1) + \Theta_{1N}] > 0$$

(iv) By comparing (10) and (23), $Var(\hat{M}_{D_0N})_{\min} > MSE(T_{i(d)N})_{\min}$ if

$$[C_{M_{yN}}^2 \lambda_N (1 - \rho_{yxN}^2) - 1 + \frac{\Theta_{1N}}{\Theta_{2N}}] > 0$$

(v) By comparing (19) and (23), $MSE(\hat{M}_{D_1N})_{\min} > MSE(T_{i(d)N})_{\min}$ if

$$[\frac{\Theta_{1N}}{\Theta_{2N}} - \frac{B_{0N}}{A_{0N}B_{0N} - C_{0N}^2 + B_{0N}}] > 0$$

(vi) By comparing (20) and (23), $MSE(\hat{M}_{D_2N})_{\min} > MSE(T_{i(d)N})_{\min}$ if

$$[\frac{\Theta_{1N}}{\Theta_{2N}} - \frac{(A_{1N}B_{1N}^2 + B_{1N}C_{1N}^2 - 2B_{1N}C_{1N}D_{1N} + B_{1N}^2 + 2B_{1N}C_{1N} - 2B_{1N}D_{1N} + B_{1N})}{A_{1N}B_{1N} - D_{1N}^2 + B_{1N}}] > 0$$

(vii) By comparing (21) and (23), $MSE(\hat{M}_{D_3N})_{\min} > MSE(T_{i(d)N})_{\min}$ if

$$[\frac{\Theta_{1N}}{\Theta_{2N}} - \frac{A_{2N}D_{2N}^2 + B_{2N}C_{2N}^2 - 2C_{2N}D_{2N}E_{2N} + 2B_{2N}C_{2N} + D_{2N}^2 - 2D_{2N}E_{2N} + B_{2N}}{A_{2N}B_{2N} - E_{2N}^2 + B_{2N}}] > 0$$

(viii) By comparing (22) and (23), $MSE(\hat{M}_{D_4N})_{\min} > MSE(T_{i(d)N})_{\min}$ if

$$[\frac{\Theta_{1N}}{\Theta_{2N}} - \frac{A_{3N}B_{3N}^2 + B_{3N}C_{3N}^2 - 2B_{3N}C_{3N}D_{3N} + B_{3N}^2 + 2B_{3N}C_{3N} - 2B_{3N}D_{3N} + B_{3N}}{A_{3N}B_{3N} - D_{3N}^2 + B_{3N}}] > 0$$

Hence, robustness is evaluated in this case to compare the proposed neutrosophic generalized estimators with other neutrosophic estimators in (1), (3), (6), (9), (11), (12), (13), and (14) to find the more effective neutrosophic median estimator. Additionally, we use real-world datasets to determine the relative effectiveness of different estimators.

Real-life application

In terms of relative efficiency, we compare the suggested family of estimators' performance to that of other competitive estimators. We chose two real-world indeterminacy interval datasets for this purpose.

Regarding relative efficiency, we compare the suggested family of estimators' performance to that of other competitive estimators. For this purpose, we chose two real-world indeterminacy interval datasets.

The first one is the Daily stock price, which is used as a neutrosophic variable because, on each day, a stock's price fluctuates between an opening price (the price during which trade begins) and a closing price (the price at which trade stops for the day). The price constantly fluctuates between a high (the highest price of the day) and a low (the lowest price), which may or may not be similar to the opening or closing price. We estimate the high and low price intervals within which the stock price falls by utilizing the daily starting price as a supplementary variable that is not a neutrosophic variable since its value is set and known for each day.

Population I Source: ⁵³ (1st June 2022 to 29th July 2022) from the link: <https://finance.yahoo.com/quote/SMSN.IL/history?p=SMSN.IL>. Y_N = Low & High prices; X_N = Opening price.

Population II Source: ⁵⁴ (1st Feb 2022 to 29th July 2022) from the link: <https://finance.yahoo.com/quote/SZKMY/history?p=SZKMY>. Y_N = Low & High prices; X_N = Opening price, where $Y_N \in (Y_L, Y_U)$ corresponds to the independent determinate variable $X_N \in (X_L, X_U)$.

Figures 2 and 3 show the trend of real-world data sets using box plots, which aids in displaying the skewness of the data. The minimum score (Lower Fence), the lower quartile, the median, the upper quartile, and the maximum score summarise data using boxplots (Upper Fence). As shown in Fig. 2, our positively skewed data suggests that the median is closer to the lower quartile. The boxplot with points outside the whiskers shows a few outliers in the data.

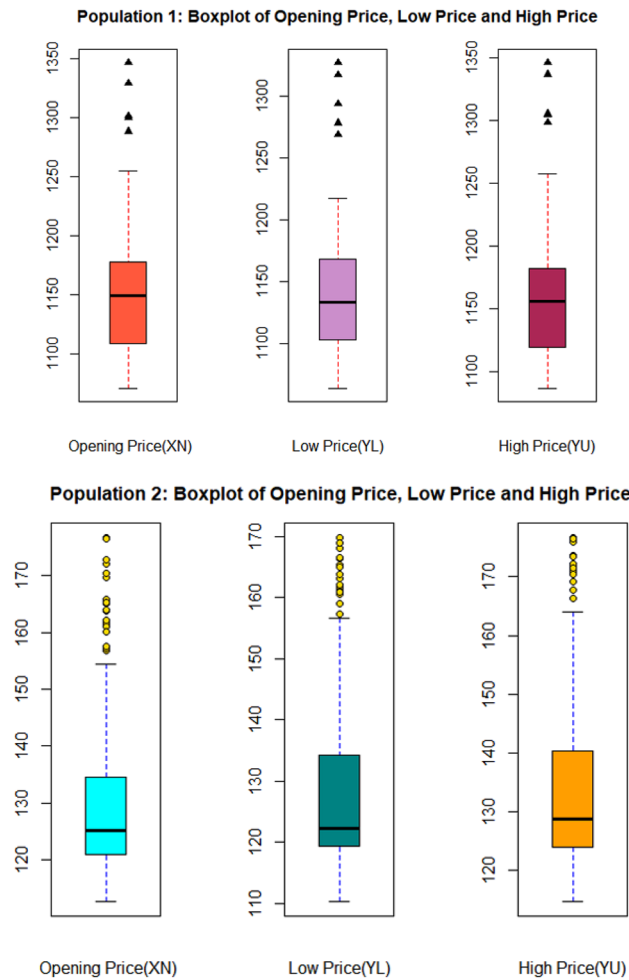


Figure 2. Boxplot of Populations I and II for each variable.

Figure 3 elaborates on the pivotal role of Q-Q (quantile–quantile) plots in statistics. These plots facilitate the graphical comparison of two probability distributions through their quantiles. They are instrumental in determining the distribution type of a random variable, spotting outliers, and assessing skewness. By plotting theoretical quantiles against sample quantiles, Q-Q plots reveal distribution traits, including skewness. Notably, deviations in the plot's upper end and a pronounced right tail indicate a right-skewed distribution, as demonstrated in standard Q-Q plot interpretations.

In addition, the following formulae are used to get the percentage relative efficiency (PRE)

$$PRE(\hat{M}_{0N}, \hat{M}_{iN}) = \frac{Var(\hat{M}_{0N})}{MSE(\hat{M}_{iN})} \times 100, \quad (\text{for } i = 0, R, E, D_0, D_1, D_2, D_3 \text{ and } D_4) \text{ and } PRE(\hat{M}_{0N}, T_{i(d)N}) = \frac{Var(\hat{M}_{0N})}{MSE(T_{i(d)N})} \times 100, \quad (\text{for } i = 1, \dots, 12).$$

Table 3 presents the complete descriptions of each population mentioned below. Table 4 presents the complete descriptions of each population mentioned below. Tables 5, 6, 7, 8, 9 and 10 elaborate the PREs of all neutrosophic estimators relative to \hat{M}_{0N} . It is observed that the PREs of $T_{i(d)N}$ estimators change with the choices of α_3 and α_4 . It is further noted that the performance of $T_{i(d)N}$ is the best among all the estimators proposed here.

As indicated by the indeterminacy interval findings from (23) for the whole data set, the neutrosophic generalized estimator $T_{i(d)N}$, is more efficient than the other suggested estimators studied. Also, the indeterminacy interval findings show that the estimator $\hat{M}_{D_{4N}}$ is superior to all other estimators except $T_{i(d)N}$ for the neutrosophic population, with a moderate or low correlation between the research variable and the supplementary variable (regardless of correlation is positive or negative).

Simulation study

We evaluate the suggested estimators' efficiency using simulated neutrosophic data, such as Y_N and X_N are neutrosophic random variates (NRV). We generate two sets of neutrosophic random numbers of $N = 1000$, which are x'_N and y'_N from neutrosophic bivariate gamma distribution using the R programming language. Additionally, motivated by the simulated population generation strategies used by ⁵⁵, we generate the

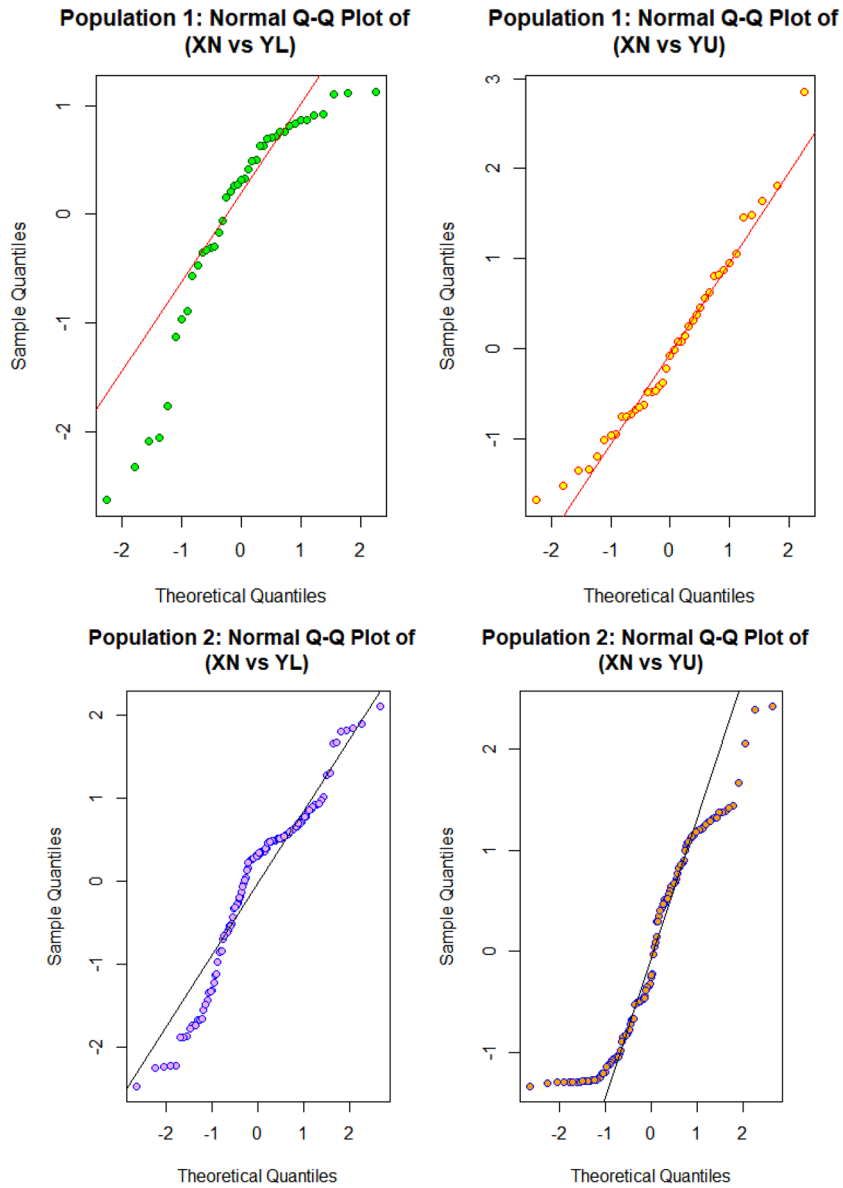


Figure 3. Normal Q-Q plot of Populations I and II for variable X_N and Y_N .

population's U transformed variables using $\rho_{yxN} = [0.7573323, 0.772941]$, $\sigma_{yN}^2 = [61.55041, 174.2082]$, $\sigma_{xN}^2 = [36.45664, 100.2741]$,

$\mu_{yN} = [21.39525, 43.65861], \mu_{xN} = [12.30978, 24.24975]$ as $Y_N = \mu_{yN} + \sigma_{yN} [\rho_{yxN}x'_N + (\sqrt{1 - \rho_{yxN}^2})y'_N]$ and $X_N = \mu_{xN} + \sigma_{xN}x'_N$.

where $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$, $y'_N \in [y'_L, y'_U]$, $Y_N \in [Y_L, Y_U]$, $\mu_{yN} \in [\mu_{yL}, \mu_{yU}]$, $\sigma_{yN}^2 \in [\sigma_{yL}^2, \sigma_{yU}^2]$, and $x'_N \in [x'_L, x'_U]$, $X_N \in [X_L, X_U], \mu_{xN} \in [\mu_{xL}, \mu_{xU}]$ and $\sigma_{xN}^2 \in [\sigma_{xL}^2, \sigma_{xU}^2]$.

Table 11 summarises the findings of the simulated data set utilized to evaluate the suggested estimators' efficiency to that of traditional estimators under neutrosophic statistics. Tables 12, 13, 14, 15 and 16 contain the percent relative efficiency of neutrosophic estimators. The analysis by simulated data also verifies that $T_{i(d)N}$, is the most efficient estimator. The simulation results suggest that the neutrosophic generalized estimator $T_{i(d)N}$, produces more accurate and precise findings than other estimators. All estimators are unbiased (up to the first order of approximation), efficient, and reliable.

In Figs. 4 and 5, we have displayed the performance of simulated data by using boxplot and Q-Q plot, respectively. The boxplots show that data is positively skewed, which implies that the median is closer to the lower or bottom quartile. The boxplot with points beyond the whiskers indicates that the data has a few outliers. Normal Q-Q plot displays that the top end of the Q-Q plot deviates from the straight line, while the lower end follows the straight line, and that the curves have a more prominent tail to the right, indicating that they are right-skewed (or positively skewed).

Values	Population I	Population II
N	41	124
n_N	[8, 8]	[45, 45]
M_{yN}	[1133, 1156.5]	[122.2155, 128.775]
M_{xN}	[1149.5, 1149.5]	[125.103, 125.103]
$f_{yN}(M_{yN})$	[0.005798982, 0.005795176]	[0.02228004, 0.0214323]
$f_{xN}(M_{xN})$	[0.005660414, 0.005660414]	[0.02137693, 0.02137693]
ρ_{yxN}	[0.9512195, 0.9512195]	[0.6774194, 0.6774194]
$C_{M_{yN}}$	[0.1522013, 0.1492065]	[0.3672466, 0.3623261]
$C_{M_{xN}}$	[0.153689, 0.153689]	[0.3739271, 0.3739271]
M_{RN}	[1208.75, 1208.75]	[144.6945, 144.6945]
Q_{DN}	[34.5, 34.5]	[6.769003, 6.769003]
Q_{AN}	[1143.5, 1143.5]	[127.7385, 127.7385]
Q_{RN}	[69, 69]	[13.53801, 13.53801]
T_{MN}	[1146.5, 1146.5]	[126.4208, 126.4208]
H_{LN}	[1142.625, 1142.625]	[124.565, 124.565]
D_{MN}	[1156.778, 1156.778]	[130.1744, 130.1744]

Table 3. Descriptive statistics of populations for single auxiliary variable.

Estimators	Population I	Population II
\hat{M}_{0N}	[100,100]	[100, 100]
\hat{M}_{RN}	[1014.094, 986.2470]	[152.1541, 149.9603]
\hat{M}_{EN}	[339.6810, 350.3235]	[175.6124, 176.3182]
$\hat{M}_{D_{0N}}$	[1050.625, 1050.625]	[184.8077, 184.8077]
$\hat{M}_{D_{1N}}$	[1050.683, 1050.681]	[184.8554, 184.8542]
$\hat{M}_{D_{2N}}$	[1050.683, 1050.681]	[184.8555, 184.8542]
$\hat{M}_{D_{3N}}$	[1050.944, 1050.946]	[184.8810, 184.8798]
$\hat{M}_{D_{4N}}$	[1052.984, 1053.050]	[184.9908, 184.9908]

Table 4. PRE's of proposed neutrosophic estimators to \hat{M}_{0N} .

Estimators	Population I	Population II
$T_{1(d)N}^\ominus$	[1055.841, 1056.150]	[185.1490, 185.1518]
$T_{2(d)N}^\ominus$	[1055.842, 1056.152]	[185.1499, 185.1527]
$T_{3(d)N}^\ominus$	[1055.841, 1056.150]	[185.1488, 185.1516]
$T_{4(d)N}^\ominus$	[1055.841, 1056.151]	[185.1491, 185.1518]
$T_{5(d)N}^\ominus$	[1055.841, 1056.151]	[185.1491, 185.1519]
$T_{6(d)N}^\ominus$	[1055.864, 1056.174]	[185.1560, 185.1588]
$T_{7(d)N}^\ominus$	[1055.842, 1056.152]	[185.1500, 185.1528]
$T_{8(d)N}^\ominus$	[1055.841, 1056.150]	[185.1486, 185.1513]
$T_{9(d)N}^\ominus$	[1055.841, 1056.151]	[185.1492, 185.1519]
$T_{10(d)N}^\ominus$	[1055.842, 1056.152]	[185.1497, 185.1525]
$T_{11(d)N}^\ominus$	[1055.841, 1056.150]	[185.1488, 185.1515]
$T_{12(d)N}^\ominus$	[1055.841, 1056.151]	[185.1491, 185.1518]

Table 5. PRE's of proposed neutrosophic estimators $T_{i(d)N}^\ominus$ to \hat{M}_{0N} .

Estimators	Population I	Population II
$T_{1(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1119, 185.1161]
$T_{2(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1125, 185.1167]
$T_{3(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1118, 185.1159]
$T_{4(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1119, 185.1161]
$T_{5(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1120, 185.1162]
$T_{6(d)N}^{\oplus}$	[1053.565, 1053.844]	[185.1165, 185.1207]
$T_{7(d)N}^{\oplus}$	[1053.555, 1053.833]	[185.1126, 185.1168]
$T_{8(d)N}^{\oplus}$	[1053.554, 1053.832]	[185.1116, 185.1158]
$T_{9(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1120, 185.1162]
$T_{10(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1124, 185.1165]
$T_{11(d)N}^{\oplus}$	[1053.554, 1053.832]	[185.1118, 185.1159]
$T_{12(d)N}^{\oplus}$	[1053.554, 1053.833]	[185.1119, 185.1161]

Table 6. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\oplus}$ to \hat{M}_{0N} .

Estimators	Population I	Population II
$T_{1(d)N}^{\otimes}$	[1056.555, 1057.150]	[185.3314, 185.3431]
$T_{2(d)N}^{\otimes}$	[1056.557, 1057.151]	[185.3327, 185.3444]
$T_{3(d)N}^{\otimes}$	[1056.555, 1057.150]	[185.3311, 185.3428]
$T_{4(d)N}^{\otimes}$	[1056.556, 1057.150]	[185.3315, 185.3431]
$T_{5(d)N}^{\otimes}$	[1056.555, 1057.150]	[185.3316, 185.3432]
$T_{6(d)N}^{\otimes}$	[1056.579, 1057.175]	[185.3412, 185.3531]
$T_{7(d)N}^{\otimes}$	[1056.557, 1057.151]	[185.3328, 185.3445]
$T_{8(d)N}^{\otimes}$	[1056.555, 1057.149]	[185.3308, 185.3424]
$T_{9(d)N}^{\otimes}$	[1056.556, 1057.150]	[185.3316, 185.3433]
$T_{10(d)N}^{\otimes}$	[1056.557, 1057.151]	[185.3324, 185.3441]
$T_{11(d)N}^{\otimes}$	[1056.555, 1057.149]	[185.3311, 185.3427]
$T_{12(d)N}^{\otimes}$	[1056.556, 1057.150]	[185.3315, 185.3432]

Table 7. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\otimes}$ to \hat{M}_{0N} .

Estimators	Population I	Population II
$T_{1(d)N}^{\circledast}$	[1064.050, 1064.946]	[185.5451, 185.5588]
$T_{2(d)N}^{\circledast}$	[1064.053, 1064.949]	[185.5475, 185.5613]
$T_{3(d)N}^{\circledast}$	[1064.050, 1064.946]	[185.5445, 185.5583]
$T_{4(d)N}^{\circledast}$	[1064.050, 1064.946]	[185.5451, 185.5589]
$T_{5(d)N}^{\circledast}$	[1064.050, 1064.946]	[185.5454, 185.5591]
$T_{6(d)N}^{\circledast}$	[1064.111, 1065.010]	[185.5639, 185.5781]
$T_{7(d)N}^{\circledast}$	[1064.053, 1064.950]	[185.5478, 185.5616]
$T_{8(d)N}^{\circledast}$	[1064.049, 1064.945]	[185.5439, 185.5576]
$T_{9(d)N}^{\circledast}$	[1064.050, 1064.946]	[185.5454, 185.5592]
$T_{10(d)N}^{\circledast}$	[1064.053, 1064.949]	[185.5470, 185.5608]
$T_{11(d)N}^{\circledast}$	[1064.049, 1064.945]	[185.5444, 185.5582]
$T_{12(d)N}^{\circledast}$	[1064.051, 1064.947]	[185.5452, 185.5590]

Table 8. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\circledast}$ to \hat{M}_{0N} .

Estimators	Population I	Population II
$T_{1(d)N}^{\odot}$	[1068.681, 1069.610]	[185.6116, 185.6235]
$T_{2(d)N}^{\odot}$	[1068.685, 1069.613]	[185.6135, 185.6254]
$T_{3(d)N}^{\odot}$	[1068.681, 1069.610]	[185.6112, 185.6231]
$T_{4(d)N}^{\odot}$	[1068.682, 1069.610]	[185.6117, 185.6236]
$T_{5(d)N}^{\odot}$	[1068.681, 1069.610]	[185.6118, 185.6237]
$T_{6(d)N}^{\odot}$	[1068.738, 1069.668]	[185.6263, 185.6385]
$T_{7(d)N}^{\odot}$	[1068.685, 1069.613]	[185.6137, 185.6256]
$T_{8(d)N}^{\odot}$	[1068.681, 1069.609]	[185.6107, 185.6226]
$T_{9(d)N}^{\odot}$	[1068.682, 1069.610]	[185.6119, 185.6238]
$T_{10(d)N}^{\odot}$	[1068.684, 1069.612]	[185.6131, 185.6250]
$T_{11(d)N}^{\odot}$	[1068.681, 1069.609]	[185.6111, 185.6230]
$T_{12(d)N}^{\odot}$	[1068.682, 1069.610]	[185.6117, 185.6236]

Table 9. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\odot}$ to \hat{M}_{0N} .

Parameters	Neutrosophic values
N	1000
n_N	[20, 20]
M_{yN}	[196.0874, 633.9646]
M_{xN}	[100.8547, 325.1492]
$f_{yN}(M_{yN})$	[0.005558005, 0.00197924]
$f_{xN}(M_{xN})$	[0.008301677, 0.002986856]
ρ_{yxN}	[0.756, 0.764]
$C_{M_{yN}}$	[0.9175533, 0.7969601]
$C_{M_{xN}}$	[1.194368, 1.029682]
M_{RN}	[180.3312, 547.1558]
Q_{DN}	[31.52584, 86.08162]
Q_{AN}	[105.1535, 335.2541]
Q_{RN}	[63.05167, 172.1632]
T_{MN}	[103.0041, 330.2016]
H_{LN}	[104.1502, 336.6766]
D_{MN}	[106.1327, 336.1194]

Table 10. Descriptive statistics for simulation study.

Estimators	PRE
\hat{M}_{0N}	[100, 100]
\hat{M}_{RN}	[137.6957, 143.8641]
\hat{M}_{EN}	[227.52, 232.4357]
$\hat{M}_{D_{0N}}$	[233.3918, 240.2091]
$\hat{M}_{D_{1N}}$	[234.4232, 240.9871]
$\hat{M}_{D_{2N}}$	[234.4415, 240.9974]
$\hat{M}_{D_{3N}}$	[235.7074, 241.9697]
$\hat{M}_{D_{4N}}$	[242.8629, 247.422]

Table 11. PRE's of proposed neutrosophic estimators to \hat{M}_{0N} .

Estimators	PRE
$T_{1(d)N}^{\ominus}$	[255.0274, 257.0464]
$T_{2(d)N}^{\ominus}$	[255.348, 257.1181]
$T_{3(d)N}^{\ominus}$	[255.0201, 257.0472]
$T_{4(d)N}^{\ominus}$	[255.0029, 257.0399]
$T_{5(d)N}^{\ominus}$	[255.0189, 257.0428]
$T_{6(d)N}^{\ominus}$	[255.3209, 257.1355]
$T_{7(d)N}^{\ominus}$	[255.343, 257.1159]
$T_{8(d)N}^{\ominus}$	[254.9967, 257.0439]
$T_{9(d)N}^{\ominus}$	[255.0344, 257.0469]
$T_{10(d)N}^{\ominus}$	[255.3343, 257.1161]
$T_{11(d)N}^{\ominus}$	[255.0043, 257.0431]
$T_{12(d)N}^{\ominus}$	[255.0187, 257.0441]

Table 12. PREs of proposed neutrosophic estimators $T_{i(d)N}^{\oplus}$ to \hat{M}_{0N} .

Estimators	PRE
$T_{1(d)N}^{\oplus}$	[253.457, 255.6164]
$T_{2(d)N}^{\oplus}$	[253.6734, 255.6645]
$T_{3(d)N}^{\oplus}$	[253.452, 255.617]
$T_{4(d)N}^{\oplus}$	[253.4404, 255.6121]
$T_{5(d)N}^{\oplus}$	[253.4513, 255.614]
$T_{6(d)N}^{\oplus}$	[253.6551, 255.6761]
$T_{7(d)N}^{\oplus}$	[253.67, 255.663]
$T_{8(d)N}^{\oplus}$	[253.4363, 255.6148]
$T_{9(d)N}^{\oplus}$	[253.4617, 255.6168]
$T_{10(d)N}^{\oplus}$	[253.6641, 255.6631]
$T_{11(d)N}^{\oplus}$	[253.4414, 255.6142]
$T_{12(d)N}^{\oplus}$	[253.4511, 255.6149]

Table 13. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\oplus}$ to \hat{M}_{0N} .

Estimators	PRE
$T_{1(d)N}^{\otimes}$	[276.5886, 272.7765]
$T_{2(d)N}^{\otimes}$	[277.1863, 272.9037]
$T_{3(d)N}^{\otimes}$	[276.5749, 272.7779]
$T_{4(d)N}^{\otimes}$	[276.5429, 272.765]
$T_{5(d)N}^{\otimes}$	[276.5728, 272.7701]
$T_{6(d)N}^{\otimes}$	[277.1358, 272.9345]
$T_{7(d)N}^{\otimes}$	[277.1769, 272.8997]
$T_{8(d)N}^{\otimes}$	[276.5314, 272.772]
$T_{9(d)N}^{\otimes}$	[276.6017, 272.7773]
$T_{10(d)N}^{\otimes}$	[277.1607, 272.9]
$T_{11(d)N}^{\otimes}$	[276.5455, 272.7706]
$T_{12(d)N}^{\otimes}$	[276.5723, 272.7723]

Table 14. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\oplus}$ to \hat{M}_{0N} .

Estimators	PRE
$T_{1(d)N}^{\otimes}$	[297.4101, 288.5158]
$T_{2(d)N}^{\otimes}$	[298.6929, 288.7828]
$T_{3(d)N}^{\otimes}$	[297.3808, 288.5187]
$T_{4(d)N}^{\otimes}$	[297.3124, 288.4917]
$T_{5(d)N}^{\otimes}$	[297.3763, 288.5024]
$T_{6(d)N}^{\otimes}$	[298.5841, 288.8476]
$T_{7(d)N}^{\otimes}$	[298.6726, 288.7745]
$T_{8(d)N}^{\otimes}$	[297.288, 288.5065]
$T_{9(d)N}^{\otimes}$	[297.4381, 288.5176]
$T_{10(d)N}^{\otimes}$	[298.6376, 288.7752]
$T_{11(d)N}^{\otimes}$	[297.318, 288.5035]
$T_{12(d)N}^{\otimes}$	[297.3753, 288.5071]

Table 15. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\oplus}$ to \hat{M}_{0N} .

Estimators	PRE
$T_{1(d)N}^{\odot}$	[302.4588, 291.9083]
$T_{2(d)N}^{\odot}$	[303.4742, 292.1171]
$T_{3(d)N}^{\odot}$	[302.4356, 291.9106]
$T_{4(d)N}^{\odot}$	[302.3813, 291.8894]
$T_{5(d)N}^{\odot}$	[302.432, 291.8977]
$T_{6(d)N}^{\odot}$	[303.3882, 292.1677]
$T_{7(d)N}^{\odot}$	[303.4582, 292.1106]
$T_{8(d)N}^{\odot}$	[302.3619, 291.901]
$T_{9(d)N}^{\odot}$	[302.481, 291.9097]
$T_{10(d)N}^{\odot}$	[303.4305, 292.1112]
$T_{11(d)N}^{\odot}$	[302.3857, 291.8987]
$T_{12(d)N}^{\odot}$	[302.4313, 291.9015]

Table 16. PRE's of proposed neutrosophic estimators $T_{i(d)N}^{\oplus}$ to \hat{M}_{0N} .

Conclusion

Our study introduced neutrosophic estimators for accurately estimating the population median in datasets with uncertain, unclear values. Through precise additional variable measurements under simple random sampling, we developed improved neutrosophic estimators, evaluated them for bias and MSE, and demonstrated their superiority. Our proposed estimators offer the advantage of modelling uncertainty and vagueness inherent in many real-world scenarios, allowing for more flexible and nuanced decision-making processes. However, their disadvantage lies in the complexity of mathematical models and computational processes required, which can lead to increased computational costs and challenges in interpretation and precise quantification. We recommend these advanced estimators for future applications and highlight the ongoing need for research to enhance estimator effectiveness for various neutrosophic data types and sampling methods. Furthermore, future work will extend to multiple sampling designs, such as systematic, successive, and double sampling.

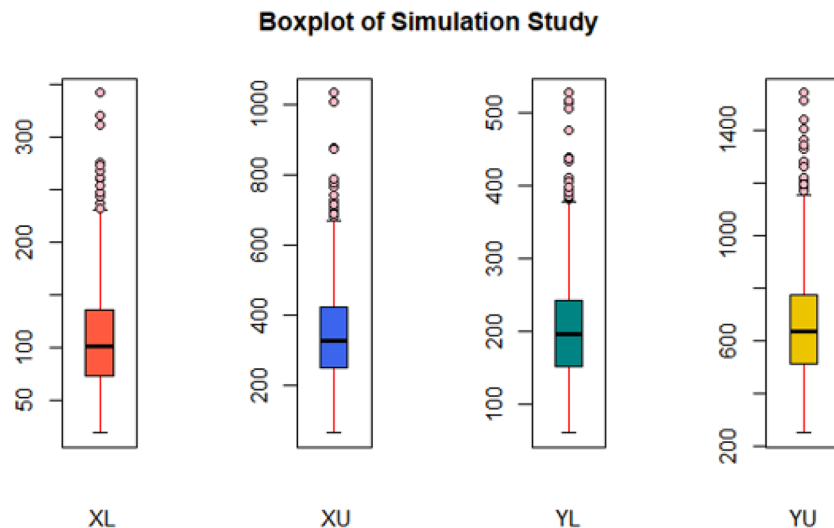


Figure 4. Boxplot of simulated data for each variable.

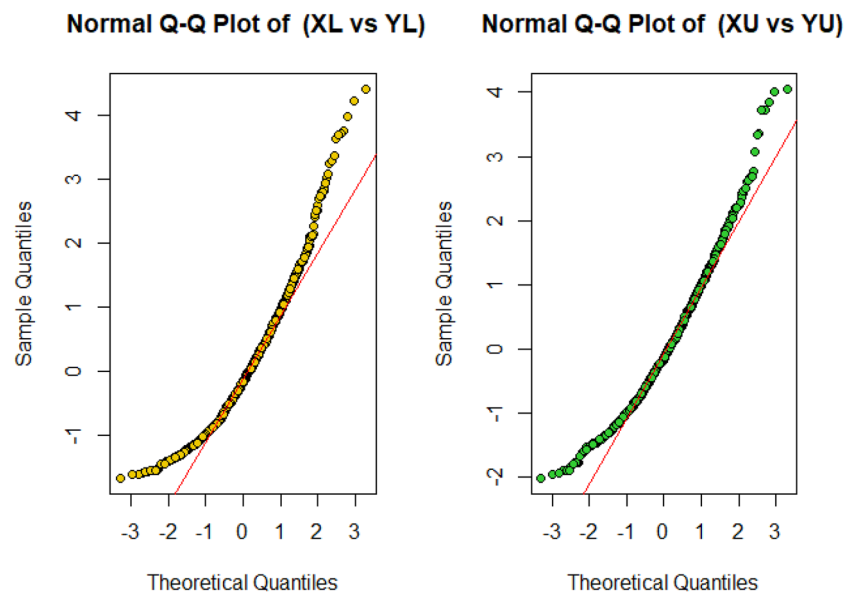


Figure 5. Normal Q-Q plot of simulated data for variable X_N and Y_N .

Data availability

Processed data are available from the corresponding author upon reasonable request.

Received: 7 November 2023; Accepted: 26 April 2024

Published online: 04 May 2024

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Author contributions

Z.M.*, S.M., B.I., J.S., and A.S. contributed to this study in the following ways: Z.M.*: Conceptualization, project administration, funding acquisition, and manuscript revision. S.M.: Data curation, formal analysis, and writing of the original draft. B.I.: Data collection, statistical analysis, and preparation of figures and tables. J.S.: Statistical methodology, code development, and review of the manuscript. A.S.: Provided expertise and insights, critically reviewed the manuscript. All authors reviewed and approved the final version of the manuscript. *Corresponding Author This statement outlines the specific contributions of each author to the manuscript, with Z.M. designated as the corresponding author.

Competing interests

The authors confirm they have no known conflicting financial interests or personal ties that might have influenced this study.

Additional information

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