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Two and three-uniform states from irredundant orthogonal arrays

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A pure quantum state of N subsystems, each with d levels, is said to be k-uniform if all of its reductions to k qudits are maximally mixed. Only the uniform states obtained from orthogonal arrays (OAs) are considered throughout this work. The Hamming distances of OAs are specially applied to the theory of quantum information. By using difference schemes and orthogonal partitions, we construct a series of infinite classes of irredundant orthogonal arrays (IrOAs), then answer the open questions of whether there exist 3-uniform states of N qubits and 2-uniform states of N qutrits, and whether 3-uniform states of qudits ($d > 2$) for high values of N can be explicitly constructed. In fact, we obtain 3-uniform states for an arbitrary number of $N \geq 8$ qubits and 2-uniform states of N qutrits for every $N \geq 4$. Additionally, we provide explicit constructions of the 3-uniform states of $N \geq 8$ qutrits, $N = 6$ and $N \geq 8$ ququarts and ququints, $N \geq 6$ qudits having d levels for any prime power $d > 6$, and $N = 8$ and $N \geq 12$ qudits having d levels for non-prime-power $d \geq 6$. Moreover, we describe an explicit construction scheme for the 2-uniform states of qudits having $d \geq 4$ levels. The proofs of existence of the 2-uniform states of $N \geq 6$ qubits are simplified by using a class of OAs. Two special 3-uniform states are obtained from IrOA(32, 10, 2, 3) and IrOA(32, 11, 2, 3) using the interaction column property of OAs.

INTRODUCTION

Multi-particle entanglement is an essential component in describing the possible quantum advantages available to metrology or information processing. The identification of multipartite quantum states with the strongest possible quantum correlations is a crucial question in quantum information theory.[1] An important open issue concerns the construction of genuinely multipartite entangled states,[2] as these have been widely applied to quantum error-correcting codes (QECCs),[3,4] teleportation,[5–8] key distribution,[9] dense coding, and quantum computation.[10] For example, in the past few years, significant development has been made in the new area of quantum machine learning, where quantum information benefits from modern information-processing technologies.[11] Quantum entanglement as a resource has been used to experimentally demonstrate various modern quantum technologies.

A pure quantum state of N subsystems with d levels is said to be k-uniform if all of its reductions to k qudits are maximally mixed.[2] An orthogonal array $OA(r, N, d_1^{n_1} d_2^{n_2} \cdots d_l^{n_l}, k)$ is an $r \times N$ matrix, having n_i columns with d_i levels, $i = 1, 2, \ldots, l$, l is an integer, $N = \sum_{i=1}^{l} n_i$, and $d_i \neq d_j$ for $i \neq j$, with the property that, in any $r \times k$ submatrix, all possible combinations of k symbols appear equally often as a row. The orthogonal array is called a mixed orthogonal array (MOA) if $l \geq 2$. Otherwise, the array is called symmetrical. Additionally, the following Rao bounds hold in OA(r, N, d, k):

$$r \geq \sum_{i=0}^{\frac{k}{2}} \binom{N}{i}(d-1)^i, \text{ if } k \text{ is even,}$$

$$r \geq \sum_{i=0}^{\frac{k-1}{2}} \binom{N}{i}(d-1)^i + \binom{N-1}{\frac{k-1}{2}}(d-1)^{\frac{k+1}{2}}, \text{ if } k \text{ is odd.}$$

OAs that achieve these bounds are called tight or saturated.[12]

An OA(r, N, d, k) is said to be an irredundant orthogonal array (IrOA) if, in any $r \times (N - k)$ subarray, all of its rows are different.[2] A link between an IrOA and a k-uniform state was established by Goyeneche et al.,[2] i.e., every column and every row of the array correspond to a particular qudit and a linear term of the state, respectively. For simplicity, the normalization factors are omitted from this paper. We summarize the link as follows:

Connection 1.1.

[2] *If* $L = \begin{pmatrix} s_1^1 & s_2^1 & \cdots & s_N^1 \\ s_1^2 & s_2^2 & \cdots & s_N^2 \\ \vdots & \vdots & \cdots & \vdots \\ s_1^r & s_2^r & \cdots & s_N^r \end{pmatrix}$ *is an* IrOA(r, N, d, k), *then the superposition of r product*

states, $|\Phi\rangle = |s_1^1 s_2^1 \ldots s_N^1\rangle + |s_1^2 s_2^2 \ldots s_N^2\rangle + \cdots + |s_1^r s_2^r \ldots s_N^r\rangle$ *is a k-uniform state.*

Many efforts have attempted to find and identify k-uniform states.[2,4] Under the generalizations of the Meyer–Wallach measure defined in,[4] k-uniform states have the largest average entanglement between blocks of qudits and the remaining elements. There has also been some progress in the construction and characterization of k-uniform states.[1,2,11,13–18] For example, Goyeneche et al.[2] constructed a 3-uniform state of six qubits and a 2-uniform state of five qubits by the judicial insertion of some minus signs. The nonexistence of the 3-uniform states of seven qubits is proved in.[14] Using the above connection, Goyeneche et al.[2] constructed 2-uniform states for an arbitrary number of $N \geq 6$ qubits using known Hadamard matrices. Yu et al.[19] constructed a 3-uniform state of eleven qubits via OAs. However, as stated in,[2] there are many open issues in multipartite quantum systems. Huber et al.[14] stated that it would be of great interest to settle the problem of whether N-qubit states exist in which all k-body reduced density are maximally mixed for $k < \lfloor \frac{N}{2} \rfloor$.

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The aim of the present study was to solve the open issues raised in[2] and the interesting question stated in[14] using IrOAs, the states related to which are useful and necessary in quantum error correction. The different uniformity of multipartite entangled states reflects different features. One motivation for searching for multipartite states of a higher uniformity is that k-uniform states of qudits offer advantages over k'-uniform states for any $0 < k' < k$. They are good for increasing the order of the density coding information rate from $d^{k'}$ to d^k.[20] It is interesting that these states form a natural extension of N-qudit Greenberger-Horne-Zeilinger states, which are 1-uniform. Moreover, a k-uniform state is also k'-uniform. As is often the case,[21–23] combinatorics can be useful to quantum information theory, and OAs are fundamental ingredients in the construction of other useful combinatorial objects.[12] The Hamming distance, difference schemes, and orthogonal partitions have been applied to many aspects of constructing OAs.[24–27] Recently, many new methods of constructing OAs of strength k, especially mixed OAs, have been presented, and many new classes of OAs have been obtained.[28–33] It is these new developments in OAs that suggest the possibility of constructing infinitely many new k-uniform states from IrOAs.

In this work, the Hamming distance and minimal distance (MD) of OAs are applied to the theory of quantum information. By using difference schemes and orthogonal partitions, we construct several infinite series of IrOAs, then consider the open issue in[2] of whether there exist 3-uniform states of N qubits and 2-uniform states of N qutrits, and whether 3-uniform states of qudits having $d > 2$ levels for high values of N can be explicitly constructed. In fact, we present several series of 3-uniform states for an arbitrary number of $N \geq 8$ qubits and 2-uniform states for an arbitrary number of $N \geq 4$ qutrits. Additionally, we provide explicit constructions of 3-uniform states of $N \geq 8$ qutrits, $N = 6$ and $N \geq 8$ ququarts and ququints, $N \geq 6$ qudits having d levels for any prime power $d > 6$, and $N = 8$ and $N \geq 12$ qudits having d levels for non-prime-power $d \geq 6$. Moreover, we explicitly construct 2-uniform states of $N \geq 4$ qudits ($d \notin \{2, 6, 10\}$), $N \geq 5$ qudits ($d = 6$), and $N = 4$ and $N \geq 6$ qudits having $d = 10$. The proofs of existence of 2-uniform states of $N \geq 6$ qubits are simplified by using a class of OAs. Furthermore, as it has been proved that IrOA(r, 7, 2, 3), IrOA(r, 7, 3, 3), IrOA(r, 6, 3, 3), IrOA(r, 4, 6, 2), IrOA(r, 6, 2, 3), IrOA(r, 4, 2, 2), and IrOA(r, 5, 2, 2) do not exist, the corresponding uniform states can not be obtained. However, two special 3-uniform states are obtained from IrOA(32, 10, 2, 3) and IrOA(32, 11, 2, 3) using the interaction column property of OAs.

The following notation, concepts, and lemmas are used in this paper.

Let A^T be the transposition of matrix A and $(\mathbf{d}) = (0, 1, \ldots, d-1)^T$. Let $\mathbf{0}_r$ and $\mathbf{1}_r$ denote the $r \times 1$ vectors of 0s and 1s, respectively. If $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{u \times v}$ with elements from a Galois field with binary operations ($+$ and $\cdot$), the Kronecker product $A \otimes B$ and the Kronecker sum $A \oplus B$ are respectively defined as $A \otimes B = (a_{ij} \cdot B)_{mu \times nv}$ and $A \oplus B = (a_{ij} + B)_{mu \times nv}$, where $a_{ij} + B$ represents the $u \times v$ matrix with entries $a_{ij} + b_{rs}$ ($1 \leq r \leq u$, $1 \leq s \leq v$). A matrix A can often be identified with a set of row vectors if necessary. Let $\lceil x \rceil$ denote the least integer not less than x, and let $\mathcal{A}$ be an additive group of d elements.

Definition 1.1. [27] Suppose that **a**, **b**, **c** are three columns in an OA(r, N, 2, 2). Then, **c** is called the interaction column between **a** and **b** if $\mathbf{c} \equiv \mathbf{a} + \mathbf{b}$ or $\mathbf{c} \equiv \mathbf{1}_r + \mathbf{a} + \mathbf{b}$ (mod 2).

Definition 1.2. [12] Let $S^l = \{(v_1, \ldots, v_l) | v_i \in S, i = 1, 2, \ldots, l\}$. The Hamming distance HD(**u**, **v**) between two vectors $\mathbf{u} = (u_1, \ldots, u_l)$, $\mathbf{v} = (v_1, \ldots, v_l) \in S^l$ is defined as the number of positions in which they differ. The minimal distance of a matrix A, written MD(A), is defined to be the minimal Hamming distance between its distinct rows. HD(A) is used to represent all the values of the Hamming distances

between two distinct rows of A. The matrix A is said to have constant Hamming distance if HD(A) is constant for any two distinct rows.

Definition 1.3. [12] An $r \times c$ matrix D with elements from $\mathcal{A}$ is called a difference scheme if it has the property that, for all i and j with $1 \leq i$, $j \leq c$, $i \neq j$, among the vector differences between the ith and jth columns, every element of $\mathcal{A}$ appears equally often. Such a matrix is denoted as $D(r, c, d)$.

For instance, if B_r is an OA(r, $c-1$, 2, 2), then $[\mathbf{0}_r, B_r]$ is $D(r, c, 2)$. $\mathcal{A}^k$, $k \geq 1$, denotes the additive group of order d^k consisting of all k-tuples of entries from $\mathcal{A}$ with the usual vector addition as the binary operation. Let $\mathcal{A}_0^k = \{(x_1, x_2, \ldots, x_k) : x_1 = \cdots = x_k \in \mathcal{A}\}$. Then, $\mathcal{A}_0^k$ is a subgroup of $\mathcal{A}^k$ of order d, and its cosets will be denoted by $\mathcal{A}_i^k$, $i = 1, \ldots, d^{k-1} - 1$.

Definition 1.4. [12] An $r \times c$ matrix D based on $\mathcal{A}$ is called a difference scheme of strength k if, for every $r \times k$ submatrix, each set $\mathcal{A}_i^k$, $i = 0$, 1, ..., $d^{k-1} - 1$, is represented equally often when the rows of the submatrix are viewed as elements of $\mathcal{A}^k$. Such a matrix is denoted by $D_k(r, c, d)$.

For $k = 2$, this definition is equivalent to Definition 1.3

Definition 1.5. [32] Let A be an OA(r, N, $d_1^{n_1} d_2^{n_2} \cdots d_l^{n_l}$, k). A set $\{A_1, A_2, \ldots, A_u\}$ of orthogonal arrays OA$\left(\frac{r}{u}, N, d_1^{n_1} d_2^{n_2} \cdots d_l^{n_l}, k_1\right)$ ($k_1 \geq 0$) is defined to be an orthogonal partition of strength k_1 of A if $\bigcup_{i=1}^u A_i = A$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. In particular, when $k_1 = 0$, A_i can be considered as an OA$\left(\frac{r}{u}, N, d_1^{n_1} d_2^{n_2} \cdots d_l^{n_l}, 0\right)$ for each $1 \leq i \leq u$.

Lemma 1.1. [12] An OA(r, N, 2, 2u) exists for $u \geq 1$ if and only if an OA(r, $N+1$, 2, 2$u+1$) exists.

Lemma 1.2. [27,34] A saturated OA(r, N, d, 2) has a constant Hamming distance $\frac{r}{d}$.

Lemma 1.3. (1) [12] If d is a prime and the integers $n \geq m \geq 1$, then a difference scheme $D(d^n, d^n, d^m)$ exists. (2) [35] If d is a prime power, then a difference scheme $D_3(d^2, d, d)$ exists. (3) [35] If $d \geq 2$, then a difference scheme $D_3(d^2, 4, d)$ exists. (4) [36] If $d \not\equiv 2 \pmod 4$, then a difference scheme $D_k(d^{k-1}, k+1, d)$ exists. In particular, $D(d, 3, d)$ exists.

Lemma 1.4. [12] (1) Let $n \geq 2$. Then an OA$\left(d^n, \frac{d^n-1}{d-1}, d, 2\right)$ exists for a prime power d. (2) If d is a prime power and $k < d$, there is an OA(d^k, $d+1$, d, k).

Lemma 1.5. [30] If B_r is an OA(r, N, d, 2) and $D(r, c, d)$ is a difference scheme, then $[B_r \oplus \mathbf{0}_r, D(r, c, d) \oplus (\mathbf{d})]$ is an OA(rd, $N+c$, d, 2). Especially, for any integer $n \geq 2$ and a prime power d, there exists an OA$\left(d^{n+1}, \frac{d^{n+1}-1}{d-1}, d, 2\right) = [B_{d^n} \oplus \mathbf{0}_d, D(d^n, d^n, d) \oplus (\mathbf{d})]$ where $B_{d^n} = $ OA$\left(d^n, \frac{d^n-1}{d-1}, d, 2\right)$.

For simplicity, we introduce further notation:

$$(A_{[1,2,\ldots,u]}, r) = \begin{pmatrix} A_1 \otimes \mathbf{1}_r \\ A_2 \otimes \mathbf{1}_r \\ \vdots \\ A_u \otimes \mathbf{1}_r \end{pmatrix} \text{ and } (r, A_{[1,2,\ldots,u]}) = \begin{pmatrix} \mathbf{1}_r \otimes A_1 \\ \mathbf{1}_r \otimes A_2 \\ \vdots \\ \mathbf{1}_r \otimes A_u \end{pmatrix} \text{ for }$$

positive integers u and r.

Lemma 1.6. [32] Let $\{A_1, A_2, \ldots, A_u\}$ be an orthogonal partition of strength k_1 of OA(r', N', d, k'), and $\{B_1, B_2, \ldots, B_v\}$ be an orthogonal partition of strength k_2 of OA(r'', N'', d, k''). Let $h = $ l.c.m.$\{u, v\}$. Then, the array $M = \left(\mathbf{1}_{\frac{h}{u}} \otimes \left(A_{[1,2,\ldots,u]}, \frac{r''}{v}\right), \mathbf{1}_{\frac{h}{v}} \otimes \left(\frac{r'}{u}, B_{[1,2,\ldots,v]}\right)\right)$ is an OA$\left(\frac{r'r''h}{uv}, N'+N'', d, k\right)$, where $k = k_1 + k_2 + 1 \leq \min\{k', k''\}$.

Lemma 1.7. [32] *Let A and B be orthogonal arrays with d rows of strength 1. Then* $C = (A \otimes \mathbf{1}_d, \mathbf{1}_d \otimes B)$ *is an orthogonal array of strength 1 and there is an orthogonal partition* $\{C_1, C_2, \ldots, C_d\}$ *of C of strength 1.*

Lemma 1.8. *If d is a prime power, there exists a difference scheme D* $(2d, 2d, d)$ *such that* $D(2d, 2d, d)^T = D(2d, 2d, d)$.

Lemma 1.9. [37] *If the transposition of a difference scheme* $D(\lambda d, \lambda d, d)$ *is still a difference scheme, then* $HD(D(\lambda d, \lambda d, d)) = \lambda d - \lambda$, *and the* $OA(\lambda d^2, \lambda d, d, 2) = D(\lambda d, \lambda d, d) \oplus (\mathbf{d})$ *has two Hamming distances* λd *and* $\lambda d - \lambda$. *Especially, if* $D(2d, 2d, d)^T = D(2d, 2d, d)$, *the* $OA(2d^2, 2d, d, 2) = D(2d, 2d, d) \oplus (\mathbf{d})$ *has two Hamming distances* $2d$ *and* $2d - 2$.

Lemma 1.10. [38] *For any positive integer* $d \geq 2$, *the following properties hold*: (1) *If* $d \notin \{2, 6\}$, *then an* $OA(d^2, 4, d, 2)$ *exists.* (2) *If* $d \notin \{2, 3, 6, 10\}$, *then an* $OA(d^2, 5, d, 2)$ *exists.* (3) *If* $d \notin \{2, 3, 4, 6, 10, 14, 18, 22\}$, *then an* $OA(d^2, 6, d, 2)$ *exists.* (4) *If* $d \notin \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\} = E$, *then an* $OA(d^2, 7, d, 2)$ *exists.* (5) *If* $d \notin E \cup \{12, 21, 28, 33, 35, 39, 42, 44, 51, 52, 54, 58, 66, 68, 74\}$, *then an* $OA(d^2, 8, d, 2)$ *exists.*

Lemma 1.11. [2] *If a k-uniform state of N qudits exists, then* $k \leq \frac{N}{2}$.

Lemma 1.12. (1) *An* $OA(r, N, d, k)$ *is irredundant if and only if its minimal distance is greater than* k. *An IrOA* (r, N, d, k) *is an IrOA* (r, N', d, k') *for* $N - k + k' \leq N' \leq N$ *with* $1 \leq k' < k$. (2) *An* $OA(r, 2k, d, k)$ *is irredundant if and only if* $r = d^k$. (3) *An* $OA(d^{k+1}, 2k + 1, d, k)$ *is irredundant if and only if it is of strength* $k + 1$. (4) *If an* $OA(r, 2k + 1, d, k)$ *is irredundant, then* $r \leq d^{k+1}$.

Lemma 1.13. *Suppose* B_r *is an* $OA(r, r - 1, 2, 2)$. *Then* $L = (\mathbf{2}) \oplus [\mathbf{0}_r, B_r]$ *is an* $OA(2r, r, 2, 3)$ *with minimal distance* $\frac{r}{2}$.

Lemma 1.14. *An* $OA(r, N, d, k)$ *with minimal distance* $w \geq k + 1$ *implies that an IrOA* (r, N', d, k) *exists for* $N - w + k + 1 \leq N' \leq N$.

Lemma 1.15. *Let* $\{A_1, A_2, \ldots, A_u\}$ *be an orthogonal partition of strength 1 of* $A = OA(r', N', d, k')$, *and* $\{B_1, B_2, \ldots, B_v\}$ *be an orthogonal partition of strength 1 of* $B = OA(r'', N'', d, k'')$ *for* $r' = du$, $r'' = dv$, $u \leq v$ *and* $k', k'' \geq 2$. *Suppose* $MD(A) = w_1$ *and* $MD(B) = w_2$. *Let* $h = l.c.m.\{u, v\}$. *If* $k', k'' \geq 3$, *the matrix* $M = \left(\mathbf{1}_{\frac{h}{u}} \otimes (A_{[1,2,\ldots,u]}, d), \mathbf{1}_{\frac{h}{v}} \otimes (d, B_{[1,2,\ldots,v]}) \right)$ *is an* $OA(d^2h, N' + N'', d, 3)$. *Otherwise, the matrix M is an* $OA(d^2h, N' + N'', d, 2)$, *and*

$$MD(M) \geq \begin{cases} \min\{w_1 + w_2, N', N''\}, & \text{if } u = v, \\ \min\{N', w_2\}, & \text{if } u|v, u < v, \\ \min\{w_1, w_2\}, & \text{otherwise.} \end{cases}$$

Lemma 1.16. *The minimal distance of an* $OA(d^k, N, d, k)$ *is* $N - k + 1$ *for* $d \geq 2$ *and* $k \geq 1$.

Lemma 1.17. *For any integer* $d \geq 2$. (1) *If* $d \notin \{2, 6\}$, *then an IrOA* $(d^2, 4, d, 2)$ *exists.* (2) *If* $d \notin \{2, 3, 6, 10\}$, *then an IrOA* $(d^2, N, d, 2)$ *exists for* $N = 4, 5$. (3) *If* $d \notin \{2, 3, 4, 6, 10, 14, 18, 22\}$, *then an IrOA* $(d^2, N, d, 2)$ *exists for* $4 \leq N \leq 6$. (4) *If* $d \notin \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\} = E$, *then an IrOA* $(d^2, N, d, 2)$ *exists for* $4 \leq N \leq 7$. (5) *If* $d \notin E \cup \{12, 21, 28, 33, 35, 39, 42, 44, 51, 52, 54, 58, 66, 68, 74\}$, *then an IrOA* $(d^2, N, d, 2)$ *exists for* $4 \leq N \leq 8$.

Lemma 1.18. *Let* $\{A_1, A_2, \ldots, A_u\}$ *be an orthogonal partition of strength 1 of* $OA(r', N', d, k')$, *and* $\{B_1, B_2, \ldots, B_u\}$ *be an orthogonal partition of strength 0 of* $OA(u, N'', d, k'')$ *for* $r' = du$ *and* $k' \geq 2$, $k'' \geq 2$. *Suppose the minimal distances of the two arrays* $OA(r', N', d, k')$ *and* $OA(u, N'', d, k'')$ *are* w_1 *and* w_2, *respectively. Then the matrix* $M = ((A_{[1,2,\ldots,u]}, 1), (d, B_{[1,2,\ldots,u]}))$ *is an* $OA(r', N' + N'', d, 2)$ *and its minimal distance satisfies* $w \geq \min\{w_1 + w_2, N'\}$.

These lemmas will form the backbone of our main theorems and results. Especially, Lemmas 1.6, 1.15, and 1.18 play an essential role in finding infinite classes of uniform states. In the 'Discussion' section, we discuss the main results and some uniform states that could have fewer terms or qudits. In the "Methods" section, we describe the central idea of the construction method. We present some examples of 2 and 3-uniform states, summarize the constructed IrOAs, and present the proofs of some lemmas in the appendix (see Supplementary Information for a detailed description of the examples, IrOAs and proofs).

RESULTS

The main results are summarized in Table 1.

Very interestingly, not only can 2 and 3-uniform states of qubits be constructed using IrOAs, but the states of high-dimensional quantum systems (qudits) can also be obtained. We now address the open issue proposed by Goyeneche et al.[2]. There is a reciprocal link between k-uniform states of N qudits and $((N, 1, k + 1))_d$ QECCs.[3] The newly constructed uniform states might be applied in quantum information protocols, quantum secret sharing,[39] and holographic codes.

Using difference schemes and orthogonal partitions, we propose a recursive construction method for OAs. By exploring the Hamming distances of OAs, especially newly constructed OAs, we explicitly construct a finite set of IrOA(r, N, d, k) for $n \leq N \leq m$. The recursive nature of the method allows us to generate infinite classes of IrOAs (see Table 1). Relying on Connection 1.1, we prove the existence of the uniform states in Table 1. Some examples are presented.

First, we describe the construction of 3-uniform states from IrOAs.

Table 1.	Existence of k-uniform states of N qudits ($d \geq 2$) and IrOA(r, N, d, k)			
k	d	k-uniform states of N qudits	IrOA(r, N, d, k)	Ref.
$k = 3$	$d = 2$	$N \geq 8$	$N = 8$ and $N \geq 10$	Th. 2.1 $N = 9$ (Supplementary Ex. 4)
	$d = 3$	$N \geq 8$	$N \geq 8$	Th. 2.4
	$d = 4, 5$	$N = 6$ and $N \geq 8$	$N = 6$ and $N \geq 8$	Th. 2.6
	$d > 6$ is a prime power	$N \geq 6$	$N \geq 6$	Th. 2.7
	$d \geq 6$ is not a prime power	$N = 8$ and $N \geq 12$	$N = 8$ and $N \geq 12$	Th. 2.8
$k = 2$	$d = 2$	$N \geq 5$	$N \geq 6$	Th. 2.11
	$d = 6$	$N \geq 5$	$N \geq 5$	Th. 2.14
	$d = 10$	$N = 4$ and $N \geq 6$	$N = 4$ and $N \geq 6$	Th. 2.16
	$d \geq 2, d \notin \{2, 6, 10\}$	$N \geq 4$	$N \geq 4$	Ths. 2.12, 2.13, 2.15 & 2.17

Construction of 3-uniform states of $N \geq 8$ qubits

By using the minimal distance and other properties of OAs, we can obtain an IrOA(r, N, 2, 3) and 3-uniform states of N qubits for $N = 8$ and $N \geq 10$. We also find a 3-uniform state of 9 qubits by adding some unimodular complex numbers in Appendix A (see Supplementary Example 4). Hence, we have 3-uniform states of N qubits for every $N \geq 8$.

Theorem 2.1. *There exists an IrOA(r, N, 2, 3) for $N = 8$ and every $N \geq$ 10 and 3-uniform states of N qubits for every $N \geq 8$.* Proof. Suppose that $n \geq 3$. It follows from Lemma 1.4 and Lemma 1.13 that an OA (2^{n+1}, 2^n, 2, 3) exists. From Lemmas 1.13 and 1.14 we can construct an IrOA(2^{n+1}, N_n, 2, 3) for $2^{n-1} + 4 \leq N_n \leq 2^n$. Obviously, $2^n + 4 \leq N_{n+1} \leq 2^{n+1}$. Therefore we can obtain $N_3 = 8$, $12 \leq N_4 \leq 16$, $20 \leq N_5 \leq 32$, etc. On the other hand, starting from the OA(12, 11, 2, 2) and using Lemma 1.5, we can obtain an OA($2^{n-1} \times 3$, $2^{n-1} \times 3 - 1$, 2, 2) for $n \geq 3$. Then an OA($2^n \times 3$, $2^{n-1} \times 3$, 2, 3) exists with the form as in Lemma 1.13. From Lemmas 1.13 and 1.14 we can also construct an IrOA($2^n \times 3$, N'_n, 2, 3) for $2^{n-2} \times 3 + 4 \leq N'_n \leq 2^{n-1} \times 3$. Thus we can respectively obtain $10 \leq N'_3 \leq 12$, $16 \leq N'_4 \leq 24$, $28 \leq N'_5 \leq 48$, and so on. And it can be easily seen that $2^{n-2} \times 3 + 4 \leq 2^n$ and $2^n + 4 < 2^{n-1} \times 3$ when $n \geq 4$. Therefore, we can obtain an IrOA(r, N, 2, 3) for $N = 8$ and $N \geq 10$. Every row of the IrOA(r, N, 2, 3) is put in kets, summed, and normalized to produce the desired 3-uniform states of N qubits for $N = 8$ and $N \geq 10$. The 3-uniform state of $N = 9$ qubits will be presented in Appendix A (see Supplementary Example 4).

This theorem suggests there is an infinite class of entangled states of N-party systems such that, tracing out any $N - 3$ subsystems, the remaining three subsystems have associated a maximally mixed state. Therefore, this plays an important role in completely settling the question on the existence of 3-uniform states of N qubits. It is also of vital importance to generalize qubits to high-dimensional quantum systems (qudits) because of the many advantages they offer over qubit-based applications.

Construction of 3-uniform states of $N \geq 8$ qutrits ($d = 3$)

By studying the minimal distance of the symmetrical OAs constructed from the orthogonal partition methods in,[32] we can obtain an IrOA(r, N, 3, 3) and 3-uniform states of N qutrits for every $N \geq 12$. By constructing OA(81, 10, 3, 3) and OA(243, 11, 3, 3) and computing their minimal distances, we have an IrOA(r, N, 3, 3) for $N = 8$, 9, 10, 11. Therefore, an IrOA(r, N, 3, 3) and 3-uniform states of N qutrits can be obtained for every $N \geq 8$.

Theorem 2.2. *There exists an IrOA(3^n, N_n, 3, 3) and 3-uniform states of N_n qutrits for $9 \cdot 2^{n-5} + 4 \leq N_n \leq 9 \cdot 2^{n-4}$ with $n \geq 5$.* Proof. First by using mathematical induction, we prove that there exists an OA (3^n, $9 \cdot 2^{n-4}$, 3, 3) having minimal distance $9 \cdot 2^{n-5}$ and an orthogonal partition $\{B_1, B_2, \ldots, B_{3^{n-1}}\}$ of strength 1 for any integer $n \geq 5$. When $n = 5$, we can obtain an OA(81, 9, 3, 3) = $D_3(27, 9, 3) \oplus (\mathbf{3})$ with minimal distance 5 from the following difference scheme:

$$D_3(27,9,3) = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_{27} \end{pmatrix} = \begin{pmatrix} 000000000000000000000000000 \\ 012200120201120012120012201 \\ 021210102102021210210102021 \\ 012200120012201120012201120 \\ 012012012120120120201201201 \\ 000222111110002222211110000 \\ 000000000111111111222222222 \\ 000111222000111222000111222 \\ 012012012012012012012012012 \end{pmatrix}^T .$$

Since $A_i = P_i \oplus (\mathbf{3})$ is an OA of strength 1 for each $1 \leq i \leq 27$, $\{A_1, A_2, \ldots, A_{27}\}$ is an orthogonal partition of strength 1 of the OA(81, 9,

3, 3). By Lemma 1.6, we can obtain an

$$\mathrm{OA}(243, 18, 3, 3) = ((A_{[1,2,\ldots,27]}, 3), (3, A_{[1,2,\ldots,27]})).$$

It follows from Lemma 1.15 that MD(OA(243, 18, 3, 3)) = 9. From Lemma 1.7, there exists an orthogonal partition $\{A_{i1}, A_{i2}, A_{i3}\}$ of strength 1 of OA ($A_i \otimes \mathbf{1}_3$, $\mathbf{1}_3 \otimes A_i$) for each $1 \leq i \leq 27$. Then there also exists an orthogonal partition $\{B_1, B_2, \ldots, B_{81}\}$ of strength 1 of the OA(243, 18, 3, 3). Assume that when $n = m$ there exists an OA (3^m, $9 \cdot 2^{m-4}$, 3, 3) with minimal distance $9 \cdot 2^{m-5}$ and an orthogonal partition $\{C_1, C_2, \ldots, C_{3^{m-1}}\}$ of strength 1. When $n = m + 1$, by induction and Lemmas 1.6 and 1.15, we can obtain an

$$\mathrm{OA}(3^{m+1}, 9 \cdot 2^{m-3}, 3, 3) = ((C_{[1,2,\ldots,3^{m-1}]}, 3), (3, C_{[1,2,\ldots,3^{m-1}]}))$$

whose minimal distance is $9 \cdot 2^{m-4}$. From Lemma 1.7, an orthogonal partition $\{C_{i1}, C_{i2}, C_{i3}\}$ of strength 1 of OA ($C_i \otimes \mathbf{1}_3$, $\mathbf{1}_3 \otimes C_i$) exists for $1 \leq i \leq 3^{m-1}$, so does an orthogonal partition $\{D_1, D_2, \ldots, D_{3^m}\}$ of strength 1 of the OA(3^{m+1}, $9 \cdot 2^{m-3}$, 3, 3). From Lemma 1.14, we can obtain an IrOA(3^n, N_n, 3, 3) for $9 \cdot 2^{n-5} + 4 \leq N_n \leq 9 \cdot 2^{n-4}$ with $n \geq 5$. The desired 3-uniform states follow from Connection 1.1.

For example, using Theorem 2.2 we can obtain an IrOA(243, N, 3, 3) for $13 \leq N \leq 18$ from the OA(243, 18, 3, 3), an IrOA(729, N, 3, 3) for $22 \leq N \leq 36$ from an OA(729, 36, 3, 3) and an IrOA(2187, N, 3, 3) for $40 \leq N \leq 72$ from an OA(2187, 72, 3, 3), etc.

Theorem 2.3. *There exists an IrOA(3^n, N'_n, 3, 3) and 3-uniform states of N'_n qutrits for $9 \cdot 2^{n-6} + 8 \leq N'_n \leq 9 \cdot 2^{n-5} + 4$ with $n \geq 6$.* Proof. We can obtain an OA(27, 4, 3, 3) = $D_3(9, 4, 3) \oplus (\mathbf{3})$ with an orthogonal partition $\{A_1, A_2, \ldots, A_9\}$ of strength 1. From Theorem 2.2, there exists an IrOA(3^{n-1}, N_{n-1}, 3, 3) for $9 \cdot 2^{n-6} + 4 \leq N_{n-1} \leq 9 \cdot 2^{n-5}$ and its an orthogonal partition $\{B_1, B_2, \ldots, B_{3^{n-2}}\}$ of strength 1 for $n \geq 6$. As $9 < 3^{n-2}$ and $9|3^{n-2}$, from Lemma 1.6 we can obtain an matrix $M = (\mathbf{1}_{3^{n-4}} \otimes (A_{[1,2,\ldots,9]}, 3), (3, B_{[1,2,\ldots,3^{n-2}]}))$. It is an OA($3^n$, N'_n, 3, 3) since $N_{n'} = N_{n-1} + 4$. By the case of $u|v$, $u < v$ in Lemma 1.15, we have MD(OA(3^n, N'_n, 3, 3)) $\geq$ min $\{4,$ MD(IrOA(3^{n-1}, N_{n-1}, 3, 3))$\} = 4$. Hence it is an IrOA(3^n, N'_n, 3, 3) for $9 \cdot 2^{n-6} + 8 \leq N_{n'} \leq 9 \cdot 2^{n-5} + 4$.

From Theorem 2.3, we can obtain an IrOA(729, N, 3, 3) for $17 \leq N \leq 22$. We can also obtain an IrOA(2187, N, 3, 3) for $26 \leq N \leq 40$, etc.

Theorem 2.4. *There exists an IrOA(r, N, 3, 3) and 3-uniform states of N qutrits for every $N \geq 8$.* Proof. By Theorems 2.2 and 2.3, we have an IrOA(3^n, N_n, 3, 3) for $9 \cdot 2^{n-6} + 8 \leq N_n \leq 9 \cdot 2^{n-4}$ with $n \geq 6$. Since $9 \cdot 2^{n-5} + 8 \leq N_{n+1} \leq 9 \cdot 2^{n-3}$ and $9 \cdot 2^{n-5} + 8 < 9 \cdot 2^{n-4}$ for $n \geq 6$, we can obtain an IrOA(r, N, 3, 3) for $N \geq 17$. Taking $n = 5$ in Theorem 2.2, we can have an IrOA(3^5, N, 3, 3) for $13 \leq N \leq 18$. By removing a column from the OA(81, 9, 3, 3) in Theorem 2.2, we can obtain an OA(81, 8, 3, 3) with minimal distance $w_2 \geq 4$. By using Lemmas 1.6 and 1.7 and the case of $u|v$, $u < v$ in Lemma 1.15, from the two arrays OA(27, 4, 3, 3) and OA(81, 8, 3, 3) we can construct an OA(243, 12, 3, 3) with minimal distance $w \geq$ min$\{4, w_2\} = 4$. So we can obtain an IrOA(243, 12, 3, 3). Taking

$$C = \begin{pmatrix} 10000111111 \\ 01011011011 \\ 00111111111 \\ 01000111222 \\ 10012012012 \end{pmatrix}_{5 \times 11},$$ by the method in[40] we have an OA

(243, 11, 3, 3) with minimal distance 4. So we can get an IrOA(243, 11, 3, 3). From the construction of an OA(d^4, $d^2 + 1$, d, 3) in,[12] we obtain an OA(81, 10, 3, 3) whose minimal distance is 6. So we can obtain an IrOA(81, N, 3, 3) for $N = 8$, 9, 10. The desired conclusion follows from Connection 1.1.

Interestingly, this theorem explicitly constructs a class of 3-uniform states that do not contain any terms with complex phase weights. And every subsystem of $N' \leq 3$ qutrits contains the same physical information. Orthogonal partitions of OAs play a key role in the proof of this theorem. Starting from a judiciously selected difference scheme and using a similar method, we obtain a general result.

Theorem 2.5. *If there exists a $D_k(r, c, d)$ with $c \geq k+1$ for $k \geq 2$ such that the minimal distance of $L = OA(rd, c, d, k) = D_k(r, c, d) \oplus$ (d) is at least $\lceil \frac{c}{2} \rceil$, then we have an IrOA($m$, N, d, 3) and 3-uniform states of N qudits for $c + 4 \leq N \leq 2c$ and every $N \geq c + 8$, and we also have an IrOA(m, N, d, 2) and 2-uniform states of N qudits for $c + 3 \leq N \leq 2c$ and every $N \geq c + 6$. Proof.* When $k \geq 3$, by using Lemmas 1.6 and 1.7 and the case of $u = v$ in Lemma 1.15, starting from L we can recursively construct a class of OAs OA(rd^{n+1}, $2^n c$, d, 3) with minimal distance $2^{n-1}c$ for any integer $n \geq 1$. Then we can obtain an IrOA(rd^{n+1}, N'_n, d, 3) for $2^{n-1}c + 4 \leq N'_n \leq 2^n c$. When $n = 1$, we have IrOA(rd^2, N, d, 3) for $c + 4 \leq N \leq 2c$. By using Lemmas 1.6 and 1.7 and the case of $u|v$, $u < v$ in Lemma 1.15, from the arrays OA(rd, 4, d, k) = $D_k(r, 4, d) \oplus$ (d) and IrOA(rd^{n+1}, N'_n, d, 3) we can construct OA(rd^{n+2}, N''_{n+1}, d, 3) for $2^{n-1}c + 8 \leq N''_{n+1} \leq 2^n c + 4$ with MD(OA(rd^{n+2}, N''_{n+1}, d, 3)) $\geq \min\{4, \text{MD}(\text{IrOA}(rd^{n+1}, N'_n, d, 3))\} = 4$ since $N''_{n+1} = N'_n + 4$. They are IrOAs. Hence an OA(rd^{n+1}, N''_n, d, 3) exists for $2^{n-2}c + 8 \leq N''_n \leq 2^{n-1}c + 4$ with $n \geq 2$. Then an OA($rd^n + 1$, N_n, d, 3) exists for $2^{n-2}c + 8 \leq N_n \leq 2^n c$ with $n \geq 2$. Since $2^{n-1}c + 8 \leq N_{n+1} \leq 2^{n+1}c$ and $2^n c \geq 2^{n-1}c + 8$, we can obtain an IrOA(m, N, d, 3) for any integer $N \geq c + 8$. Similarly, when $k \geq 2$, we also have an IrOA(m, N, d, 2) and 2-uniform states of N qudits for $c + 3 \leq N \leq 2c$ every $N \geq c + 6$.

Based on Theorem 2.5, we further consider the cases of qudits ($d \geq 4$).

Construction of 3-uniform states of $N = 6$ or $N \geq 8$ ququarts and ququints ($d = 4, 5$)

By analyzing the minimal distances of the symmetrical OAs constructed from Theorem 2.5, we obtain two infinite classes of IrOA(r, N, 4, 3) and IrOA(r, N, 5, 3) and 3-uniform states of N ququarts and ququints for $N = 6$ and every $N \geq 8$.

Theorem 2.6. *There exists an IrOA(r, N, d, 3) for $d = 4, 5$ and 3-uniform states of N ququarts and ququints for $N = 6$ and every $N \geq 8$. Proof.* Consider the case $d = 4$. By Lemma 1.16 we can obtain the OA(64, 6, 4, 3) = D_3(16, 6, 4) $\oplus$ (4) with minimal distance 4, where

$$D_3(16,6,4) = \begin{pmatrix} 0000000000000000 \\ 0123230132101032 \\ 0231132020133102 \\ 0231201331021320 \\ 0000111122223333 \\ 0123012301230123 \end{pmatrix}^T.$$

Then an IrOA(64, N, 4, 3) exists for $N = 6$. It follows from Theorem 2.5 that starting from the D_3(16, 6, 4) we have an IrOA(r, N, 4, 3) for $10 \leq N \leq 12$ and every $N \geq 14$. Now we prove the existence of an IrOA(r, N, 4, 3) for $N = 8, 9, 13$. From the OA(64, 6, 4, 3) we can easily have two arrays OA(64, 5, 4, 3) and OA(64, 4, 4, 3) with minimal distances 3 and 2. We proceed our proof by using Lemmas 1.6, 1.7 and 1.15. From the OA(64, 4, 4, 3) we can obtain OA(256, 8, 4, 3). From the arrays OA(64, 4, 4, 3) and OA(64, 5, 4, 3) we can obtain OA(256, 9, 4, 3). Moreover, from the two arrays OA(64, 5, 4, 3) and OA(256, 8, 4, 3) we can construct an OA(1024, 13, 4, 3). They all have minimal distances ≥ 4. Then three arrays IrOA(256, 8, 4, 3), IrOA(256, 9, 4, 3) and IrOA(1024, 13, 4, 3) exist. Therefore, we have an IrOA(r, N, 4, 3) and 3-uniform states of N

ququarts for $N = 6$ and every $N \geq 8$. Similarly, when $d = 5$, from the D_3(25, 6, 5) in[35] we also have an IrOA(r, N, 5, 3) and 3-uniform states of N ququints for $N = 6$ and every $N \geq 8$.

Construction of 3-uniform states of N qudits with $d \geq 6$ levels

Starting from the difference schemes $D_3(d^2, d, d)$ and $D_3(d^2, 4, d)$ respectively and exploring the minimal distances of OAs, we will construct an IrOA(r, N, d, 3) and the 3-uniform states of N qudits for every $N \geq 6$ with any prime power $d > 6$ levels, and an IrOA(r, N, d, 3) and the 3-uniform states of N qudits for $N = 8$ and every $N \geq 12$ with any non-prime-power $d \geq 6$ levels.

Theorem 2.7. *For any prime power $d > 6$, there exists an IrOA(r, N, d, 3) and 3-uniform states of N qudits for every $N \geq 6$. Proof.* By Lemmas 1.4 and 1.16, we can get an OA(d^3, $d + 1$, d, 3) with minimal distance $d-1$, so we have an IrOA(d^3, N, d, 3) for $6 \leq N \leq d + 1$ from Lemma 1.14. It follows from Lemmas 1.3 and 1.16 that a difference scheme $D_3(d^2, d, d)$ exists and the minimal distance of OA(d^3, d, d, 3) = $D_3(d^2, d, d) \oplus$ (d) is $d - 2$. Then an OA(d^3, N_3, d, 3) exists with minimal distance ≥ 2 for $4 \leq N_3 \leq d$. From the OA(d^3, N_3, d, 3) ($4 \leq N_3 \leq d$) and Lemmas 1.6 and 1.7 and the case of $u = v$ in Lemma 1.15, we can construct an OA(d^4, N, d, 3) with minimal distance $d \geq \{2 + 2, 4, 4\} = 4$. They are all IrOA($d^4$, N, d, 3) for $8 \leq N \leq 2d$. By Theorem 2.5 and from the $D_3(d^2, d, d)$, we can obtain an IrOA(r, N, d, 3) for every $N \geq d + 8$. Since $7 < d + 1$ and $d + 7 \leq 2d$, we can obtain an IrOA(r, N, d, 3) for every $N \geq 6$. From Connection 1.1, we can obtain an IrOA(r, N, d, 3) and 3-uniform states of N qudits for every $N \geq 6$.

Using similar arguments, we can construct 3-uniform states with any non-prime-power $d \geq 6$ levels.

Theorem 2.8. *For any non-prime-power $d \geq 6$, there exists an IrOA(r, N, d, 3) and 3-uniform states of N qudits for $N = 8$ and every $N \geq 12$. Proof.* From Lemma 1.3, a difference scheme $D_3(d^2, 4, d)$ exists for any $d \geq 2$. It follows from Lemma 1.16 that the minimal distance of OA(d^3, 4, d, 3) = $D_3(d^2, 4, d) \oplus$ (d) is 2. Therefore, we have an IrOA(r, N, d, 3) for $N = 8$ and every $N \geq 12$ by Theorem 2.5. By Connection 1.1, we can obtain 3-uniform states of N qudits for $N = 8$ and every $N \geq 12$.

The existence of maximally entangled states lies at the intersection of quantum theory and discrete mathematics, and determining their mathematical structure is extremely technical. The characterization and classification of such entanglement have now become issues of great interest. The constructed states with higher dimensions offer an increase in the channel capacity for quantum communication, and provide a higher error rate tolerance and enhanced security in quantum key distribution.[41] High-dimensional entanglement offers great potential for applications in quantum information, particularly in quantum communications.

However, the states of higher dimensions are significantly different from the trivial extension of those of lower dimensions. A key problem in the construction of 3-uniform states of a non-prime-power dimension is the shortage of proper mathematical tools. For example, in non-prime-power dimensions the Galois fields do not exist. Therefore, studying these states is even more challenging than in the case of prime-power dimensions. In practice, the available experimental data are usually quite limited, such that devising effective ways for characterizing high-dimensional entanglement is very challenging. Interestingly, the states of any dimensions can be consistently obtained using the proposed methods.

It is interesting that 3-uniform states are a natural extension of 1-uniform states. They are useful for increasing the order of density coding information rate from d^1 to d^3.[20] Moreover, a 3-uniform state is also 2-uniform.

We continue to use our method for 2-uniform states.

New results of construction of 2-uniform states of $N \geq 5$ qubits
For $N \geq 6$, there exist 2-uniform states of N qubits that can be constructed from two infinite classes of Hadamard matrices based on the Hadamard matrices H_2 and H_{12} of orders 2 and 12.[2] However, we simplify the construction of these 2-uniform states using only the known class of orthogonal arrays OA(2^n, 2^n-1, 2, 2) with $n \geq 3$.

Theorem 2.9. *A saturated orthogonal array* OA($r, r-1, 2, 2$) *for $r \geq 8$ implies the existence of an* IrOA($r, N, 2, 2$) *and 2-uniform states of N qubits for $\frac{r}{2} + 2 \leq N \leq r - 1$. Proof.* It follows from Lemma 1.2 that the OA($r, r-1, 2, 2$) has constant Hamming distance $\frac{r}{2}$. Then we have an IrOA($r, N, 2, 2$) for $\frac{r}{2} + 2 \leq N \leq r - 1$ from Lemma 1.14. By Connection 1.1, we can obtain 2-uniform states of N qubits for $\frac{r}{2} + 2 \leq N \leq r - 1$.

Example 2.1. By Theorem 2.9, we can obtain an IrOA($8, N, 2, 2$) and 2-uniform states of N qubits for $6 \leq N \leq 7$, an IrOA($12, N, 2, 2$) and 2-uniform states of N qubits for $8 \leq N \leq 11$ and an IrOA($20, N, 2, 2$) and 2-uniform states of N qubits for $12 \leq N \leq 19$ from OA($8, 7, 2, 2$), OA($12, 11, 2, 2$) and OA($20, 19, 2, 2$), respectively.

Theorem 2.10. *If $B_{\frac{r}{2}}$ is an* OA$\left(\frac{r}{2}, \frac{r}{2} - 1, 2, 2\right)$ *for $r \geq 16$ and $8|r$, then the array $L = [B_{\frac{r}{2}} \oplus \mathbf{0}_2, (\mathbf{0}_{\frac{r}{2}}, B_{\frac{r}{2}}) \oplus (\mathbf{2})]$ allows us to obtain an* IrOA($r, N_r, 2, 2$) *for $\frac{r}{4} + 3 \leq N_r \leq r - 1$. Proof.* If $B_{\frac{r}{2}}$ is an OA, then L is an OA ($r, r-1, 2, 2$) by Lemma 1.5. Let $L_1 = B_{\frac{r}{2}} \oplus \mathbf{0}_2$, $L_2 = (\mathbf{0}_{\frac{r}{2}}, B_{\frac{r}{2}}) \oplus (\mathbf{2})$. It follows from Lemma 1.2 that the Hamming distances HD, HD$_1$, HD$_2$ of L, L_1, L_2 statisfy HD $= \frac{r}{2}$, HD$_1 \leq \frac{r}{4}$, HD$_2 \geq \frac{r}{4}$. Consequently, we can obtain an IrOA($r, N_r, 2, 2$) for $\frac{r}{4} + 3 \leq N_r \leq r - 1$ by respectively deleting $j\left(0 \leq j \leq \frac{3}{4}r - 4\right)$ columns from the L. In fact, if $0 \leq j \leq \frac{r}{2} - 1$, by deleting arbitrary j columns in L_1, we can obtain an IrOA($r, N_r, 2, 2$) for $\frac{r}{2} \leq N_r \leq r - 1$. If $\frac{r}{2} - 1 < j \leq \frac{3}{4}r - 4$, by deleting all columns in L_1 and arbitrary $j - \left(\frac{r}{2} - 1\right)$ columns in L_2, we can obtain an IrOA($r, N_r, 2, 2$) for $\frac{r}{4} + 3 \leq N_r < \frac{r}{2}$. We complete the proof.

Example 2.2. By Theorem 2.9 and an OA($24, 23, 2, 2$), we can obtain an IrOA($24, N, 2, 2$) and 2-uniform states of N qubits for $14 \leq N \leq 23$. However, from the OA($24, 23, 2, 2$) satisfying the condition in Theorem 2.10 we can obtain more IrOA($24, N, 2, 2$) with $9 \leq N \leq 13$ than from Theorem 2.9. Thus we have $9 \leq N \leq 23$. Hence more 2-uniform states can be obtained.

Theorem 2.11. *An* IrOA($r, N, 2, 2$) *for $N \geq 6$ can be obtained from a known class of* OA($2^n, 2^n - 1, 2, 2$) *with $n \geq 3$ and 2-uniform states of N qubits exist for every $N \geq 5$. Proof.* In Theorem 2.10, let $B_{2^n} = $ OA($2^n, 2^n - 1, 2, 2$) which exists for every $n \geq 3$ from Lemma 1.5. Then we can obtain an IrOA($2^{n+1}, N_{2^{n+1}}, 2, 2$) for $2^{n-1} + 3 \leq N_{2^{n+1}} \leq 2^{n+1} - 1$, which does not include the Hadamard conjecture. So we have $2^n + 3 \leq N_{2^{n+2}} \leq 2^{n+2} - 1$. And it can be easily proved that $2^{n+1} - 1 > 2^n + 3$ when $n \geq 3$. Then we can get an IrOA($r, N, 2, 2$) for every $N \geq 7$. In particular, when $n = 3$, using the OA($8, 7, 2, 2$) again we have IrOA($8, 6, 2, 2$) by Theorem 2.9 Therefore, there exists an IrOA($r, N, 2, 2$) for every $N \geq 6$. On the other hand, we have a 2-uniform state $|\Psi_5\rangle$ of $N = 5$ qubits as follows:. $|\Psi_5\rangle = |11111\rangle + |00110\rangle + |01100\rangle + |10101\rangle - |10000\rangle + |01001\rangle - |00011\rangle + |11010\rangle$. Another expression of a 2-uniform state of five qubits can been found in.[2]. Hence, there exist 2-uniform states of N qubits for $N \geq 5$. We finish the proof.

Example 2.3. By Theorem 2.9, we can obtain an IrOA($16, N, 2, 2$) and 2-uniform states of N qubits for $10 \leq N \leq 15$ from any OA($16, 15, 2, 2$). However, from the OA($16, 15, 2, 2$) satisfying the condition in Theorem 2.11 we can obtain more IrOA($16, N, 2, 2$) and 2-uniform states of N qubits than for $7 \leq N \leq 15$ from Theorem 2.9.

Remark 1. Let $n = 4$ in Theorem 2.11 Then we have IrOA($32, 31, 2, 2$) for $11 \leq N_{32} \leq 31$. Further, we can obtain $9 \leq N_{32}' \leq 31$ by using the property of interaction columns (see Example 2.4).

Example 2.4. The existence of an IrOA($32, N, 2, 2$) *and 2-uniform states of N qubits for $9 \leq N \leq 31$.*

By Theorem 2.9, we can obtain an IrOA($32, N, 2, 2$) and 2-uniform states of N qubits for $18 \leq N \leq 31$ from any OA($32, 31, 2, 2$). However, from the OA($32, 31, 2, 2$) satisfying the condition in Theorem 2.11 we can obtain IrOA($32, N, 2, 2$) and 2-uniform states of N qubits for $11 \leq N \leq 31$. Let $B_{16} = $ OA($16, 15, 2, 2$) $= (\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1; \mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2; \dots; \mathbf{a}_5, \mathbf{b}_5, \mathbf{c}_5)$ and OA($32, 31, 2, 2$) $= [B_{16} \oplus \mathbf{0}_2, (\mathbf{0}_{16}, B_{16}) \oplus (\mathbf{2})]$. Then the Hamming distance of the OA $[\mathbf{0}_{16}, B_{16}] \oplus (\mathbf{2})$ is greater than or equal to 8. And the OA($16, 15, 2, 2$) has 6 columns ($\mathbf{a}_4, \mathbf{b}_4, \mathbf{c}_4; \mathbf{a}_5, \mathbf{b}_5, \mathbf{c}_5$) such that $\mathbf{c}_i$ is the interaction between $\mathbf{a}_i$ and $\mathbf{b}_i$ for $i = 4, 5$. The Hamming distance between the rows in the columns ($\mathbf{a}_4, \mathbf{b}_4, \mathbf{c}_4; \mathbf{a}_5, \mathbf{b}_5, \mathbf{c}_5$) is less than or equal to 4. Then by respectively removing ($\mathbf{a}_4, \mathbf{b}_4, \mathbf{c}_4; \mathbf{a}_5, \mathbf{b}_5, \mathbf{c}_5$) $\oplus (\mathbf{2})$ and ($\mathbf{c}_3; \mathbf{a}_4, \mathbf{b}_4, \mathbf{c}_4; \mathbf{a}_5, \mathbf{b}_5, \mathbf{c}_5$) $\oplus (\mathbf{2})$ from $[\mathbf{0}_{16}, B_{16}] \oplus (\mathbf{2})$, one can obtain two arrays IrOA($32, 10, 2, 2$) and IrOA($32, 9, 2, 2$).

High-dimensional entanglement is different from two-level entanglement. Furthermore, general quantum states could be multilevel. They break the generic classical constraints and provide a new perspective for quantum mechanics.

Construction of 2-uniform states of $N \geq 4$ qutrits
We can derive an IrOA($r, N, 3, 2$) and 2-uniform states of N qutrits for every $N \geq 4$ by the property and the minimal distances of OAs and difference schemes.

Theorem 2.12. *There exists an* IrOA($r, N, 3, 2$) *and 2-uniform states of N qutrits for every $N \geq 4$. Proof.* We first prove that there exists an IrOA($3^{n+1}, N_n, 3, 2$) for $3^{n-1} + 3 \leq N_n \leq \frac{3^{n+1}-1}{2}$ with $n \geq 1$. When $n = 1$, any OA($9, 4, 3, 2$) is an IrOA($3^2, N_1, 3, 2$) satisfying $4 \leq N_1 \leq 4$. When $n \geq 2$, Lemmas 1.3 and 1.4 imply the existence of OA$\left(3^n, \frac{3^n-1}{2}, 3, 2\right)$ and $D(3^n, 3^n, 3)$. For a given n, let $B_{3^n} = $ OA$\left(3^n, \frac{3^n-1}{2}, 3, 2\right)$, $L_1 = B_{3^n} \oplus \mathbf{0}_3$ and $L_2 = D(3^n, 3^n, 3) \oplus (\mathbf{3})$. Then, $L = [L_1, L_2]$ is an OA$\left(3^{n+1}, \frac{3^{n+1}-1}{2}, 3, 2\right)$ from Lemma 1.5. It follows from Lemma 1.2 that the Hamming distances HD, HD$_1$, and HD$_2$ of L, L_1, and L_2 satisfy HD $= 3^n$, HD$_1 \leq 3^{n-1}$ (HD$_1 = 0, 3^{n-1}$), HD$_2 \geq 2 \cdot 3^{n-1}$. Consequently, we can obtain an IrOA($3^{n+1}, N_n, 3, 2$) for $3^{n-1} + 3 \leq N_n \leq \frac{3^{n+1}-1}{2}$ by deleting $0 \leq j \leq \frac{7}{2}(3^{n-1} - 1)$ columns from L. In fact, by deleting any $0 \leq j \leq \frac{3^n-1}{2}$ columns in L_1, we obtain an IrOA($3^{n+1}, N_n, 3, 2$) for $3^n \leq N_n \leq \frac{3^{n+1}-1}{2}$. For $\frac{3^n-1}{2} < j \leq \frac{7}{2}(3^{n-1} - 1)$, deleting all the columns in L_1 and any $j - \left(\frac{3^n-1}{2}\right)$ columns in L_2 gives IrOA($3^{n+1}, N_n, 3, 2$) for $3^{n-1} + 3 \leq N_n < 3^n$. It can easily be shown that $3^{n-1} + 3 \leq N_n \leq \frac{3^{n+1}-1}{2}$, $3^n + 3 \leq N_{n+1} \leq \frac{3^{n+2}-1}{2}$, and $3^n + 3 < \frac{3^{n+1}-1}{2}$ for $n \geq 2$. When $n = 2, 3, 4, \dots$, we can respectively obtain an IrOA($3^{n+1}, N_n, 3, 2$) for $6 \leq N_2 \leq 13$, $12 \leq N_3 \leq 40$, $30 \leq N_4 \leq 121$, and so on. Thus, we can obtain an IrOA($r, N, 3, 2$) for $N = 4$ and every $N \geq 6$. When $N = 5$, taking

$$\text{OA}(18, 7, 3, 2) = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_7) = \begin{pmatrix} 012012012012012012 \\ 012012120201120201 \\ 012120012201201120 \\ 012201201012120120 \\ 012120201120012201 \\ 012201120120201012 \\ 000000111111222222 \end{pmatrix}^T,$$

we can obtain an IrOA($18, 5, 3, 2$) $= (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_5)$ that leads to a 2-

uniform state of five qutrits. By Connection 1.1, we can obtain 2-uniform states of N qutrits for every $N \geq 4$.

For these states, every matrix in which the density is reduced to two qutrits is proportional to the identity I_{3^2}. This theorem settles the question of the 2-uniform states of N qutrits.[2] Note that the OAs L in this theorem can be recursively constructed starting from OA(9, 4, 3, 2).

Construction of 2-uniform states of $N \geq 4$ qudits with a prime power $d \geq 4$ levels

For a prime power $d \geq 4$, we can obtain an IrOA(r, N, d, 2) and 2-uniform states of N qudits for every $N \geq 4$ from the property and minimal distances of OAs and some special difference schemes.

Theorem 2.13. *For a prime power $d \geq 4$, there exists an IrOA(r, N, d, 2) and 2-uniform states of N qudits for an arbitrary number of $N \geq 4$. Proof.* For a prime power d and an integer $n \geq 2$, Lemmas 1.3, 1.4, and 1.5 imply that there exists an OA

$$L = \text{OA}\left(d^{n+1}, \frac{d^{n+1}-1}{d-1}, d, 2\right) = [B_{d^n} \oplus \mathbf{0}_d, D(d^n, d^n, d) \oplus (\mathbf{d})]$$

where $B_{d^n} = \text{OA}\left(d^n, \frac{d^n-1}{d-1}, d, 2\right)$. Let $L_1 = B_{d^n} \oplus \mathbf{0}_d$ and $L_2 = D(d^n, d^n, d) \oplus (\mathbf{d})$. Then, $L = [L_1, L_2]$. It follows from Lemma 1.2 that the Hamming distances HD, HD_1, and HD_2 of L, L_1, and L_2 satisfy $\text{HD} = d^n$, $HD_1 \leq d^{n-1}$, and $HD_2 \geq (d-1) \cdot d^{n-1}$. Consequently, we can obtain an IrOA(d^{n+1}, N_n, d, 2) for $d^{n-1} + 3 \leq N_n \leq \frac{d^{n+1}-1}{d-1}$ by deleting $0 \leq j \leq \frac{1}{d-1}[(d^2 - d + 1)d^{n-1} - 3d + 2]$ columns from L in the same way as in Theorem 2.12. When $n = 1$, we can obtain IrOA(d^2, N_1, d, 2) for $4 \leq N_1 \leq d + 1$ from a known OA(d^2, $d+1$, d, 2). When $n \geq 2$, we have $d^{n-1} + 3 \leq N_n \leq \frac{d^{n+1}-1}{d-1}$, $d^n + 3 \leq N_{n+1} \leq \frac{d^{n+2}-1}{d-1}$. It can easily be proved that $d^n + 3 \leq \frac{d^{n+1}-1}{d-1}$, so we obtain IrOA($r$, N, d, 2) for $N \geq d + 3$. To cover the gap $N = d + 2$, consider the difference scheme $D(2d, 2d, d)$ satisfying $D(2d, 2d, d)^T = D(2d, 2d, d)$, which exists by Lemma 1.8. Let $L' = D(2d, 2d, d) \oplus (\mathbf{d})$. Then, L' is an OA. It follows from Lemma 1.9 that L' has two Hamming distances, $2d-2$ and $2d$. Hence, we can obtain IrOA($2d^2$, N'_d, d, 2) for $5 \leq N'_d \leq 2d$ from Lemma 1.14. As $d + 1 \geq 5$ and $2d > d + 3$, there exists an IrOA(r, N, d, 2) for $N \geq 4$, which leads to the desired 2-uniform states.

Example 2.5. Let $d = 4$. When $n = 1, 2, 3, \ldots$, it follows from Theorem 2.13 that we have an IrOA(4^2, N_1, 4, 2) for $4 \leq N_1 \leq 5$, an IrOA(4^3, N_2, 4, 2) for $7 \leq N_2 \leq 21$, an IrOA(4^4, N_3, 4, 2) for $19 \leq N_3 \leq 85$, and so on. In addition, we can obtain an IrOA(32, N'_4, 4, 2) for $5 \leq N'_4 \leq 8$. Therefore, there exists an IrOA(r, N, 4, 2) and 2-uniform states of N ququarts for every $N \geq 4$.

Construction of 2-uniform states of N qudits with a non-prime-power $d \geq 4$ levels

By computing the minimal distance of symmetrical OAs constructed from the orthogonal partition methods in,[32] we derive the following results.

When $d = 6$, there exists an IrOA(r, N, 6, 2) and 2-uniform states of N qudits for $N \geq 5$.

When $d > 6$ ($d \not\equiv 2 \pmod 4$) is not a prime power, we can obtain an IrOA(r, N, d, 2) and 2-uniform states of N qudits for $N \geq 4$.

When $d > 6$ ($d \equiv 2 \pmod 4$) is not a prime power, since every IrOA(r, N', d, 3) is also an IrOA(r, N, d, 2), we can obtain an IrOA(r, N, d, 2) and 2-uniform states of N qudits for $N = 7$, 8 and $N \geq 11$ from IrOA(r, N', d, 3) for $N' = 8$ and $N' \geq 12$ in Theorem 2.8. We also have an IrOA(r, N, d, 2) for $N = 4$, 6, 9, 10. Then, there exists an IrOA(r, N, d, 2) and 2-uniform states of N qudits for $N = 4$ and $N \geq 6$. In particular, when $d \geq 14$ ($d \equiv 2 \pmod 4$) is not a prime power, there exists an IrOA(d^2, 5, d, 2) and 2-uniform states of N qudits for $N \geq 4$.

Theorem 2.14. *There exists an IrOA(r, N, 6, 2) and 2-uniform states consisting of N six-level systems for every $N \geq 5$. Proof.* We can obtain an OA(72, 6, 6, 2) = $D(12, 6, 6) \oplus (\mathbf{6})$ in which $D(12, 6, 6)$ is given by.[42] As the minimal distance of the OA(72, 6, 6, 2) is 4, we can generate two arrays, IrOA(72, 6, 6, 2) and IrOA(72, 5, 6, 2). By,[43] $L = [(\mathbf{6}) \oplus \mathbf{0}_{12}, D(12, 6, 6) \oplus (\mathbf{6})]$ is an OA with minimal distance at least 4. Thus, we have an IrOA(72, 7, 6, 2). Using Lemmas 1.6 and 1.7 and the case $u = v$ in Lemma 1.15, OA(72, 5, 6, 2) and any three of its columns, say OA(72, 5, 6, 2) and OA(72, 3, 6, 2), give an OA(432, 8, 6, 2) with minimal distance 3. Thus, we have IrOA(432, 8, 6, 2). Additionally, by Theorem 2.5 and from $D(12, 6, 6)$, we obtain an IrOA(r, N, 6, 2) for every $N \geq 9$. Therefore, the desired IrOAs and 2-uniform states exist.

Theorem 2.15. *For any non-prime-power $d > 6$ and $d \not\equiv 2 \pmod 4$, there exists an IrOA(r, N, d, 2) and 2-uniform states of N qudits for every $N \geq 4$. Proof.* By Lemma 1.3, a difference scheme $D(d, 3, d)$ exists for any $d \geq 3$ and $d \not\equiv 2 \pmod 4$ and a $D_3(d^2, 4, d)$ exists for any $d \geq 2$. Take $L = D(d, 3, d) \oplus (\mathbf{d})$. It follows from Lemma 1.16 that $\text{MD}(L) = 2$. Using the $D(d, 3, d)$ and Theorem 2.5, we obtain an IrOA(r, N, d, 2) for any integer $N \geq 9$. By Lemma 1.17(3), we have three arrays IrOA(d^2, 6, d, 2), IrOA(d^2, 5, d, 2), and IrOA(d^2, 4, d, 2). From OA(d^3, 4, d, 3) = $D_3(d^2, 4, d) \oplus (\mathbf{d})$, IrOA($d^2$, 4, d, 2), and Lemma 1.18, we obtain an OA(d^3, 8, d, 2) with minimal distance at least 4. Therefore, we have two arrays IrOA(d^3, 8, d, 2) and IrOA(d^3, 7, d, 2). Thus, we have an IrOA(r, N, d, 2) for $N \geq 4$. The desired 2-uniform states follow from Connection 1.1.

Theorem 2.16. *For any non-prime-power $d > 6$ and $d \equiv 2 \pmod 4$, there exists an IrOA(r, N, d, 2) and 2-uniform states of N qudits for $N = 4$ and $N \geq 6$. Proof.* It follows from Theorem 2.8 that an IrOA(r, N', d, 3) exists for $N' = 8$ and $N' \geq 12$. From Lemma 1.12, we obtain an IrOA(r, N, d, 2) for $N = 7,8$ and $N \geq 11$. By Lemmas 1.3 and 1.16, an $D_3(d^2, 4, d)$ exists for any $d \geq 2$. Thus, we have an OA(d^3, 4, d, 3) = $D_3(d^2, 4, d) \oplus (\mathbf{d})$ with minimal distance 2. From the OA(d^3, 4, d, 3) and the case $u = v$ in Lemma 1.15, we can construct an OA(d^4, 8, d, 3) with minimal distance at least 4. Furthermore, from the OA(d^4, 8, d, 3) and the OA(d^3, 4, d, 3) and Lemma 1.18 we have an OA(d^4, 12, d, 2) with minimal distance at least 6. Hence we can obtain an IrOA(d^4, N, d, 2) for $9 \leq N \leq 12$. From the two arrays OA(d^3, 4, d, 3) and OA(d^2, 2, d, 2) and Lemma 1.18, we have an OA(d^3, 6, d, 2) with minimal distance at least 3. Hence, it is an IrOA(d^3, 6, d, 2). It follows from Lemma 1.17(1) that the IrOA(d^2, 4, d, 2) exists. Thus, we have the desired results.

Theorem 2.17. *For any non-prime-power $d \geq 14$ and $d \equiv 2 \pmod 4$, there exists an IrOA(r, N, d, 2) and 2-uniform states of N qudits for every $N \geq 4$. Proof.* This is an immediate consequence of Lemma 1.17(2) and Theorem 2.16.

Entanglement is an outstanding property of multiparty quantum systems that has been identified as an essential component in many fields of the theory of quantum information. In recent years, there has been considerable interest in generating entangled states of high dimensions. Such states can, in principle, contain a great deal of entanglement, which makes them more complicated but offer promising perspectives for quantum applications. Therefore, they are important to quantum communication.

We constructed 3-uniform states before 2-uniform states because we needed to use some results from the former case. That the newly constructed 3-uniform states are 2-uniform follows immediately from the strength of their corresponding IrOAs. We hope that our method for 2- and 3-uniform states will also contribute to demonstrating the existence of k-uniform states for $k \geq 4$ in the homogeneous case.

Table 2.	Existence of 2-uniform states of N subsystems with $d \geq 2$ level			
N	$2 \leq N \leq 3$	4	5	$N \geq 6$
d				
$d = 2$	–	?	b	a
$d = 3$	–	a	a	a
$d = 4$	–	a	a	a
$d = 5$	–	a	a	a
$d = 6$	–	?	a	a
$d = 7$	–	a	a	a
$d = 8$	–	a	a	a
$d = 9$	–	a	a	a
$d = 10$	–	a	?	a
$d \geq 11$	–	a	a	a

Table 3.	Existence of 3-uniform states of N subsystems with $d \geq 2$ levels							
N	$2 \leq N \leq 5$	6	7	8	9	10	11	$N \geq 12$
d								
$d = 2$	–	?	?	a	b	a	a	a
$d = 3$	–	?	?	a	a	a	a	a
$d = 4$	–	a	?	a	a	a	a	a
$d = 5$	–	a	?	a	a	a	a	a
$d > 6$ is a prime power	–	a	a	a	a	a	a	a
$d \geq 6$ is not a prime power	–	?	?	a	?	?	?	a

DISCUSSION

The present work not only answers the question proposed by Goyeneche et al.[2] and provides a positive answer to the interesting problem stated by Huber et al.,[14] but also presents more general results. In Tables 2 and 3, for any given $d \geq 2$, we summarize the precise existence of the states for every value of N. In the tables, "a" denotes that a state can be constructed from an IrOA, whereas "b" represents that a state can be constructed with some judicially inserted minus signs. The symbol "–" means that such a state cannot exist by Lemma 1.11, whereas "?" indicates that we have not yet solved the problem.

Tables 2 and 3 indicate that for any given $d \geq 2$ (prime power or non-prime-power), we exhaustively solve the problem regarding an explicit construction of the 2 and 3-uniform states of every N qudits from IrOAs, except for at most five values of N. They are closely related to QECCs, multi-unitary permutation matrices, and mutually orthogonal quantum Latin squares and cubes.[1] Furthermore, for the k-uniform states with symbol "?" in Tables 2 and 3, we consider the following five cases:

(1) The nonexistence of 3-uniform states of seven qubits and 2-uniform states of four qubits has already been proved.[2,13] In Theorem 3.1, we prove the nonexistence of IrOA(r, 7, 2, 3) and IrOA(r, 4, 2, 2).

(2) The existence of 3-uniform states of six qubits and 3-uniform states of N qutrits for $N = 6, 7$ has been presented in,[44] whereas the nonexistence of IrOA(r, 6, 2, 3), IrOA(r, 6, 3, 3), and IrOA(r, 7, 3, 3) has been proved in Theorem 3.1.

(3) Interestingly, the existence of 2-uniform states of four qudits ($d = 6$) is unknown. It cannot be derived from IrOAs, because there is no IrOA(r, 4, 6, 2) by Theorem 3.1. However, 2-uniform states of five qudits ($d = 6$) can be constructed from IrOA(72, 5, 6, 2) by Theorem 2.14.

(4) The 3-uniform state of seven ququints exists,[44] although the existence of IrOA(r, 7, 5, 3) is unknown.

(5) The existence of IrOA(r, 7, 4, 3), IrOA(r, 5, 10, 2), and IrOA(r, N, d, 3) for $N = 6, 7, 9, 10, 11$ and non-prime-power $d \geq 6$ and their corresponding states remains unresolved.

Theorem 3.1. IrOA(r, 7, 2, 3), IrOA(r, 7, 3, 3), IrOA(r, 6, 3, 3), IrOA(r, 4, 6, 2), IrOA(r, 6, 2, 3), IrOA(r, 4, 2, 2), and IrOA(r, 5, 2, 2) do not exist. Proof. Assume that there exists an IrOA(r, 7, 2, 3). Then, $r \leq 16$ by Lemma 1.12. However, this contradicts the fact that neither OA(16, 7, 2, 4) nor OA(r, 7, 2, 3) exists for $r < 16$ (ref. [45]). By the same argument, IrOA(r, 7, 3, 3), IrOA(r, 6, 3, 3), and IrOA(r, 4, 6, 2) do not exist. Assume there exists an IrOA(r, 7, 3, 3). Then, $r \leq 81$, but this contradicts the fact that neither OA(81, 7, 3, 4) nor OA(r, 7, 3, 3) exists for $r < 81$. Assume that IrOA(r, 6, 3, 3) and IrOA(r', 4, 6, 2) exist. Then, $r = 27$ and $r' = 36$, but this contradicts the fact that neither OA(27, 6, 3, 3) nor OA(36, 4, 6, 2) exists. Similarly, IrOA(r, 6, 2, 3), IrOA(r, 4, 2, 2) and IrOA(r, 5, 2, 2) do not exist.

It is tempting to believe that some of the states obtained in this paper will be good for experimental purposes and help to quantify[46] the level of entanglement in some multipartite system. The states are also ideal candidates for quantum information protocols and quantum secret sharing. It is expected that remarkable progress will be made in the field of QECCs by application of the results presented herein. As stated in,[2] there are many open issues to solve regarding the construction and characterization of entanglement in multipartite quantum systems. The results presented in this paper will establish a foundation for solving other open problems, such as the construction of k-uniform states of N qudits ($d \geq 2$) for $k \geq 4$, including the problem stated by Huber et al.,[14] and heterogeneous multipartite systems,[17] since the proposed construction methods can be suitable for IrOAs of any strength $k \geq 4$ and irredundant mixed orthogonal arrays (IrMOAs).[17,32,33]

In the construction process, we often encounter the problem that some uniform states could have fewer terms or qudits. If the Hadamard conjecture is considered, Theorems 2.9 and 2.10 state that the number of terms in many 2-uniform states could be reduced. Moreover, the following three theorems indicate that some 2-uniform states can have fewer terms. Furthermore, from Lemmas 1.13 and 1.14, there exist some 3-uniform states with fewer terms than those obtained by Theorem 2.1 (for instance, the two 3-uniform states obtained from IrOA(40, 16, 2, 3) and IrOA(48, 16, 2, 3) by Lemmas 1.13, 1.14, and Theorem 2.1, respectively). Additionally, the states obtained by Theorem 3.5 have fewer qubits.

Theorem 3.2. There exist 2-uniform states of N qudits with 10 levels for $N = 6, 7, 9, 10$ that have fewer terms than the states given by Theorem 2.16. Proof. Let OA(200, 6, 10, 2) = [D(20, 5, 10) ⊕ **(10)**, **(10)** ⊕ **0**$_{20}$], where D(20, 5, 10) can be found in.[42] Then, the minimal distance of the OA is 3. Therefore, it is an IrOA(200, 6, 10, 2). By exchanging the 16th and 17th rows of the D(40, 6, 10) in,[42] we obtain the difference scheme D'(40, 6, 10) and an OA(400, 7, 10, 2) = [D'(40, 6, 10) ⊕ **(10)**,**(10)** ⊕ **0**$_{40}$] with minimal distance 3. Hence, this is IrOA(400, 7, 10, 2). Utilizing Lemmas 1.6 and 1.7 and the case $u = v$ in Lemma 1.15, we can construct OA(2000, 10, 10, 2) with minimal distance 4 from the OA(200, 5, 10, 2) = D(20, 5, 10) ⊕ **(10)** with minimal distance 2. Then we obtain an IrOA(2000, 10, 10, 2) and an IrOA(2000, 9, 10, 2). However, from Theorem 2.16, we can only obtain IrOA(10^3, 6, 10, 2), IrOA(10^4, 7, 10, 2), IrOA(10^4, 9, 10, 2), and IrOA(10^4, 10, 10, 2). Thus, the theorem holds.

Theorem 3.3. For $N = 6, 9, 10$, there are 2-uniform states of N qudits with 14 levels that have fewer terms than those given by Theorem 2.16. Proof. By exchanging the 26th and 27th rows in the D(28, 5, 14) from,[42] we obtain the difference scheme D'(28, 5, 14) and an

OA(392, 6, 14, 2) = [D'(28, 5, 14) $\oplus$ **(14)**,**(14)** $\oplus$ **0**$_{28}$] with minimal distance 3. Hence, this is IrOA(392, 6, 14, 2). Furthermore, from an OA(392, 5, 14, 2) = D(28, 5, 14) $\oplus$ **(14)** with minimal distance 2, Lemmas 1.6 and 1.7, and the case $u = v$ in Lemma 1.15, we can construct an OA(5488, 10, 14, 2) with minimal distance 4. Then, we obtain IrOA(5488, 10, 14, 2) and IrOA(5488, 9, 14, 2). These IrOAs have fewer rows than IrOA(14^3, 6, 14, 2), IrOA(14^4, 9, 14, 2), and IrOA(14^4, 10, 14, 2) obtained by Theorem 2.16. Therefore, the result follows from Connection 1.1.

Theorem 3.4. *Some uniform states obtained by Lemma 1.17 have fewer terms than those given by Theorems 2.15 and 2.16 for $N = 7, 8$. Proof.* This follows from the fact that some IrOA(d^2, N, d, 2) obtained by Lemma 1.17 have fewer rows than that given by Theorems 2.15 and 2.16 for $N = 7, 8$.

Remark 2. The following theorem and Theorems 2.10 and 2.11 indicate that we can find more IrOA(n, N, 2, 2) than that suggested by Theorem 2.9 using the construction methods for OA(n, $n - 1$, 2, 2). Therefore, we can consider the lowest value of N such that k-uniform states exist for fixed r, d, and k.

Theorem 3.5. *If an OA(r, $r - 1$, 2, 2) is obtained from an MOA $(r, l + u, d_1^1 \cdots d_l^1 2^u, 2)$ by using the expansive replacement method and OA(d_i, $d_i - 1$, 2, 2), where $d_i \geq 4$ for $i = 1, \ldots, l$, then for any $1 \leq j \leq l$, there exists an OA(r, $r - 1 - [(d_1 - 1) + \cdots + (d_j - 1)]$, 2, 2) with a minimal distance that is at least $\frac{1}{2}[r - (d_1 + d_2 + \cdots + d_j)]$. Proof.* It follows from Lemma 1.2 that the Hamming distances of the OA(r, $r - 1$, 2, 2), OA(d_i, $d_i - 1$, 2, 2) ($i = 1, \ldots, l$) are $\frac{r}{2}$, $\frac{d_i}{2}$, respectively, since they are saturated. For any $1 \leq j \leq l$, deleting the corresponding $d_1 - 1 + d_2 - 1 + \cdots + d_j - 1$ columns obtained by the expansive replacement method in[12] from OA(r, $r - 1$, 2, 2), we can obtain the desired OA(r, $r - 1 - [(d_1 - 1) + \cdots + (d_j - 1)]$, 2, 2).

For instance, if B_{32} is an OA(32, 31, 2, 2), then from the OA(64, 63, 2, 2) = [$B_{32} \oplus$ **0**$_2$,(**0**$_{32}$, B_{32}) $\oplus$ **(2)**] and using Theorem 2.10, we can obtain an IrOA(64, N, 2, 2) for $19 \leq N \leq 63$. However, from an OA(64, 21, $8^7 2^{14}$, 2) and Theorem 3.5 and by deleting the seven 8-level columns, we have OA(64, 14, 2, 2) with minimal distance at least 4. Thus, an IrOA(64, N, 2, 2) exists for $13 \leq N \leq 63$.

In the future, we will study the optimal problems of the uniform states from IrOAs. At first, we will investigate the optimal problems of IrOAs, as they play an important role in uniform states. For some given N, d, k, we consider an IrOA(r, N, d, k) in which r attains its minimum, while we can also consider the lowest value of N such that an OA(r, N, d, k) exists for fixed r, d, k. For some given r, N, d, we consider an IrOA(r, N, d, k) in which k attains its maximum, and we can also consider the highest value of d such that an OA(r, N, d, k) exists for fixed r, N, k.

METHODS

By means of the construction methods of OAs, we can investigate the properties of the required IrOAs (see "Introduction" section). Using difference schemes, orthogonal partitions, and the Hamming distance, we creatively construct several infinite series of IrOAs. From these IrOAs, we then obtain the 2 and 3-uniform states of almost every N qudits for any given number of levels $d \geq 2$ by means of the link between an IrOA and a k-uniform state, which was established by Goyeneche et al.[2] In particular, a key construction of uniform states of non-prime-power dimensions is due to the orthogonal partition method for recursively constructing OAs, which is new development in combinatorial designs (see 'Results' section). Starting with some special difference schemes and IrOAs, or exploring the expansive replacement method, we can then obtain some uniform states having fewer terms or qudits (see 'Discussion' section).

AUTHOR CONTRIBUTIONS

S.Q.P. proposed the initial idea and goal. S.Q.P. and X.Z. performed the theoretical calculations and contributed to writing the manuscript. S.Q.P., X.Z. and X.L. discussed and analyzed the results. Q.J.Z. programmed to calculate Hamming distances of OAs.

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