Insights through dimensions

Dimensional analysis is a powerful tool for assessing physical problems, reaffirms **Tina Hecksher**.

If presented with an equation where time and length are added, we immediately know that the equation cannot be correct — it is deeply encoded in a physicist's DNA that it is only meaningful to add terms of the same dimension. However, we are allowed to multiply or divide different quantities and obtain new — derived — quantities in doing so. For instance, if length l is divided by time t we obtain a new quantity, l/t, velocity. Which quantities are considered fundamental and which are derived is a matter of convention and convenience rather than a law of nature[1,2].

In physics classes it is taught to include units when evaluating formulas with values and to always check that the units work out right in equations. That way we easily detect if we forgot to square a term somewhere or the numbers we plugged into our equation were stated in different units and need a numerical correction. Although perhaps not appreciated as such by many (non-physicist) scientists, this is the simplest application of dimensional analysis.

A more advanced use is rewriting equations in terms of the characteristic quantities of the physical situation described, sometimes referred to as 'non-dimensionalizing' equations. This is useful when assessing which terms in a differential equation are important. A simple example of this is the equation describing the free fall of a body: $\mathrm{d}^2z/\mathrm{d}t^2 = g$, where g is the gravitational acceleration felt near the Earth's surface and z is the height of the body. There are two initial values: the body's initial velocity v_0 and its initial height z_0. Defining dimensionless variables in terms of these parameters, $\tilde{z} = z/z_0$ and $\tilde{t} = t/(z_0/v_0)$, the differential equation reads $\mathrm{d}^2\tilde{z}/\mathrm{d}\tilde{t}^2 = (gz_0)/(v_0^2)$, where the right-hand side is dimensionless. Writing the equation this way we see that there is really only one parameter in the problem; doubling the initial velocity may be counterbalanced by increasing z_0 by a factor of four to give the exact same trajectory in dimensionless units. In fact, this dimensionless number is an instance of the so-called Froude number used in fluid dynamics, $\mathrm{Fr} = v_0/\sqrt{gz_0}$.

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The most powerful use of dimensional analysis is for predicting how the outcome of an experiment depends on the variables and at the same time providing theoretical insight. The recipe for doing this is the following: make a list of all quantities on which the answer must depend, then write down the dimensions of these quantities, and finally demand that these quantities be combined into a functional form that provides the right dimension. This scheme was cast into a formal framework by Buckingham in 1921 and is often referred to as the Buckingham π-theorem[3].

Dimensional arguments have been used by some of the greatest physicists. In his seminal paper on the model of the atom explaining the absorption spectra of hydrogen, Niels Bohr (pictured) based his derivation on a dimensional argument[4]: "By the introduction of this quantity [Planck's constant] the question of the stable configuration of the electrons in the atoms is essentially changed, as this constant is of such dimensions and magnitude that it, together with the mass and charge of the particles, can determine a length of the order of magnitude required." Bohr noted that electrodynamics alone could not predict the size of an atom. But introducing Planck's constant provided the right size — the Bohr radius.

If we were to derive the Bohr radius from dimensional analysis today, we would argue that the relevant physical quantities are the elementary charge e and the vacuum permittivity ε_0, because the atom involves interacting charges; the mass of the electron m_e, because it is the electron that is orbiting the much heavier nucleus; and finally Planck's constant h, because we know that on the small scale of the atom energy is quantized. These quantities can be combined in a unique way to provide a length: $a = C(\varepsilon_0 h^2)/(e^2 m_e)$. Dimensional analysis provides the answer up to a dimensionless constant C, but setting $C = 1$ and plugging in the numbers we arrive at $a = 1.7$ Å.

Rayleigh was another enthusiastic advocate of dimensional analysis (which he called 'the principle of similitude') and provided many more examples of physical insight obtained through dimensional analysis[5]. But the concept is not limited to physics: there is a neat proof of the Pythagorean theorem due to Einstein that relies on dimensional reasoning[6].

Dimensional analysis may come across as simply trying to fit pieces of a puzzle by trial and error. However, identifying the quantities that are relevant for a given problem is a demanding task that requires deep physical insight. So, 'for dimensional reasons' is a valid argument in physics, and dimensional analysis truly deserves a place in any physicist's toolbox. ❐

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