MATHEMATICS

A topological tie-in

Simon Willerton

A CENTURY ago, physicists such as Lord Kelvin[1] first delved into the theory of knots in an attempt to explain the Periodic Table, the idea at the time being that knots represented some form of stability, and that joining two knots could represent

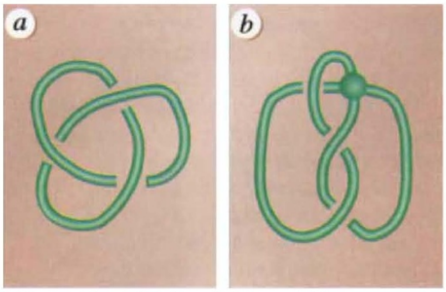

FIG. 1*a*, The (right) trefoil knot. This has Jones polynomial $1+x+x^3-x^4$. *b*, A knot with a double point or self-intersection.

the fusing of atoms. When this got nowhere, knot theory was left as an esoteric subject of study for pure mathematicians. But unexpected discoveries led to a renaissance in the 1980s; explanations for these have been trickling in from various areas, one being singularity theory, a branch of abstract mathematics, and another being quantum physics, where knots on the sub-atomic level were once again being considered. These two approaches have now been synthesized by Bar-Natan in a paper[2] to appear in *Topology*.

Mathematicians consider an abstraction of a knot in a piece of string that has had its ends joined to stop it being untied, so a knot is a knotted circle in three-dimensional space that does not intersect itself (Fig. 1*a*)[3]. The study of knots is part of topology, the mathematical discipline in which shapes can be deformed in any way provided that they don't tear. Two knots are said to be of the same knot type if one can be deformed into the shape of the other.

But how do mathematicians tackle such problems as "Make a list (albeit infinite) of all of the possible types of knots" and "Determine (by a finite algorithm) whether two knots are of same type"? One approach is to study knot invariants, which assign to each knot a more tractable mathematical object, say a polynomial or a number, that is the same when evaluated on two knots of the same type. (A particular invariant may have the same value on two knots of different type.)

Until the mid-1980s, most knot invariants were derived from well understood topological ideas, but then an important invariant, the Jones polynomial[4], was defined combinatorially (in other words, one draws a two-dimensional picture of the knot, then uses an algorithm to cut it into sections and thus obtain the invariant. The final step is to show that it doesn't matter what drawing was originally taken). This is a powerful invariant, in that it is good at distinguishing between knots, but at the time it lacked the topological description required if it was to be placed in a more natural context, alongside previous invariants.

Evidently, a fresh approach was needed. Vassiliev[5], a specialist in singularity theory working in Moscow, looked at knots not as knotted circles in three-dimensional space, but as points in the abstract space of all knots. Analogously, consider a quadratic function: this can be thought of as a 'u' or 'n' shaped graph, or as being defined by the equation $f(x) = ax^2+bx+c$, where a, b and c are real numbers. So each such function can be thought of as the point (a,b,c) in the space of all quadratics. The u-shaped graphs correspond to the triples (a,b,c) with $a > 0$ and the n-shaped graphs $a < 0$. These two regions are separated by a 'wall' — the plane $a = 0$, corresponding to functions with straight-line graphs.

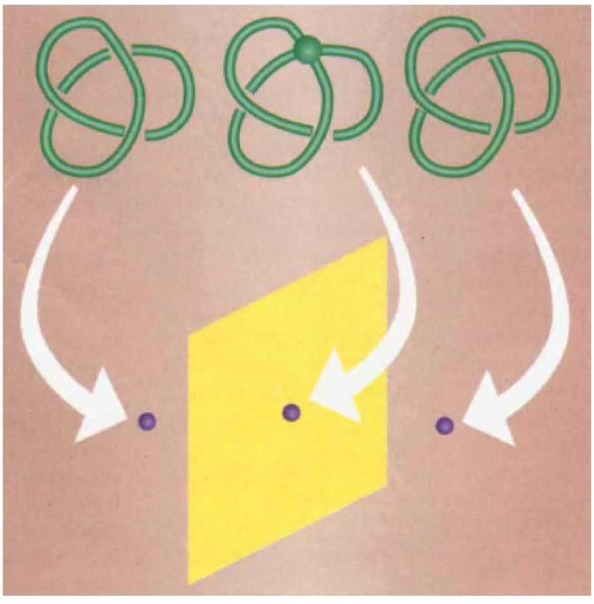

FIG. 2 A section of wall in knot space. On the right we have the trefoil knot, on the wall we have a knot with a double point and on the left a 'knot' which is unknotted. Notice how in passing through the wall we push one strand of the knot through another.

RÉSUMÉ

Seven up

THE number of examples of the earliest known bird, *Archaeopteryx*, inches towards double figures with the description of a seventh fossil (P. Wellnhofer, *Archaeopteryx* **11**, 1–30; 1993). Although all the specimens are usually thought to belong to one species, *A. lithographica*, a degree of variation hints at greater taxonomic diversity. In 1985, for example, Howgate removed the smallest ('Eichstatt') specimen to a new taxon, *Jurapteryx recurva*. Wellnhofer treads the same path in his description of the latest specimen, which yields much new information, particularly about about the jaws and sternum. The former betray a distinctive theropod heritage, the latter the unmistakable signs of an able flyer. But more telling is its growth: although the animal was small (larger only than the Eichstatt specimen), the sternum is fully ossified, quite unlike that of the larger 'Berlin' specimen. This prompts Wellnhofer to place it in a new species, *A. bavarica*, which will make the feathers fly at the International Ornithological Congress in Vienna this August.

Storm warning

IN the search for correlations between volcanic activity and other phenomena (see opposite page), L. G. Mastin has investigated yet another possibility (*Geol. Soc. Am. Bull.* **106**, 175–185; 1994). Mastin reports that almost all of 28 'explosion-like' seismic events documented at Mount St Helens, Washington, in 1989–91 happened hours to days after storms in the area. The suggestion has been made before, but Mastin sets a figure on it; the probability that the unusually rainy weather before each eruption occurred by chance is, he says, a few per cent at most. He speculates that gas trapped beneath a shallow impermeable cap is vented after a storm because of landslides or the faster growth of cooling fractures.

Small change

S. M. BARNS and co-workers have dipped into Jim's Black Pool in Wyoming and used molecular phylogenetic techniques to open another window on biodiversity (*Proc. natn. Acad. Sci. U.S.A.* **91**, 1609–1613; 1994). The pool is a hot spring in Yellowstone National Park, and Barns *et al.* were interested in the archaebacteria to be found in its sediments. They extracted DNA from sediment and, using the polymerase chain reaction, ended up with ribosomal DNA sequences which they compared with sequences in a database. The number and variety of clones recovered, especially of a subgroup known as the Crenarchaeota, is remarkable — remarkable, say the authors, to the extent that the phylogenetic organization of the archaebacteria, as understood from culture techniques, will have to be re-evaluated.

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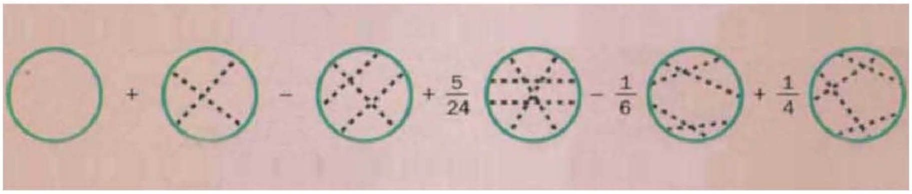

FIG. 3 The first six terms in an expansion of the trefoil knot in terms of chord diagrams. Each chord shows which parts of the circle are joined at double points (imagine shrinking the chord until the sides of the circle meet at its centre).

Similarly, a knot corresponds to a point in the space of all knots. But whereas the function space was three-dimensional in the quadratic case, the space of all knots has infinite dimensions, making it (to put it mildly) harder to visualize. In this space the different knot types are separated into chambers by walls, the walls consisting of knots that have a self-intersection or 'double point' (Figs 1*b* and 2 show examples). The walls meet at 'corners' which correspond to knots with two self-intersections. This analysis of walls and corners then generalizes to knots with a greater number of self-intersections.

Vassiliev approached the study of these walls by studying numerical knot invariants (invariants that assign a number to every knot, and thus a single number to each 'chamber' described above). Such an invariant can be extended to knots with a single self-intersection — in other words, its value can be defined on the walls — by looking at the knots on either side of the wall. The number on the wall is simply the difference between the values on either side. Then this procedure can be extended to knots with two double points and so on. Vassiliev invariants are those invariants that are zero when extended to knots with a certain (large) number of double points. These invariants are quite simple to understand from certain viewpoints, and Birman and Lin[6] proved that the Jones polynomial is built up from invariants of this type.

The other approach was from quantum physics where one considers the probability of events happening: the probability of 'observables' such as the position and momentum of particles taking certain values. Around the mid-1980s physicists started considering space-times with no notion of distance. Think of two points marked on a rubber sheet: it is impossible to define a distance between the points, as stretching the sheet would alter the distance. Without the notion of distance, position and momentum become meaningless, so one needs topological ways of describing the motion of particles: in fact the only sensible thing to consider as an observable is the knottedness of the particles' trajectories.

A tool known as the Feynman path integral can be used to calculate the probabilities of particles traversing knots in three-dimensional space-time, these probabilities depending only on the type of knot involved. A near-miraculous discovery was made by Witten[7], a physicist at Princeton, when he was able to construct a model for space-time in which this probabilistic invariant generalized the Jones polynomial. However, although physicists happily use the Feynman integral (as it seems to give them sensible answers), it has never been properly defined mathematically. A more rigorous analysis of Witten's construction has been sorely needed.

And this is where Bar-Natan's work comes in. Working under Witten, Bar-Natan[2,8] used perturbative methods to expand the Witten integral as a sum of terms that are easily understood mathematically, rather like expanding e^x as powers of x. This expansion is in terms of generalized Feynman diagrams (see Fig. 3); the integral describes all possible ways that the particle can behave, and each Feynman chord diagram corresponds to one of these types of behaviour. Bar-Natan has shown that the invariant derived from each chord diagram is a Vassiliev invariant, neatly uniting the two topological approaches.

Where next for the knot theorists? Kontsevich[9] has constructed an integral that expands every knot as a sum of chord diagrams, and has proved that these methods give rise to all invariants of Vassiliev's type. His integral was originally impossible to calculate except for the simplest knots, but mathematicians are now finding ways of making it more calculable. So the latest results, although they have not answered every question, have certainly increased the arsenal with which knot theorists can attack their favourite tangles. □

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DAEDALUS

Less is more

HOMOEOPATHY is a well-established fringe therapy. It offers paradoxical medicines whose effectiveness rises as their dose is reduced. The manufacturers seem curiously unconvinced by their own claim — they usually recommend a half dose for children, never the double dose required by the theory. But many formulations are so dilute that they are statistically unlikely to contain a single molecule of the active ingredient, so the precise dosage can hardly matter.

What chemical effects intensify with dilution? One candidate, says Daedalus, is electrochemical: the equilibrium potential on an electrode in solution. The Nernst equation asserts that this potential grows monotonically more negative as the concentration of the solution declines. Ultimately, zero concentration should give an infinite negative voltage. This extreme, of course, can never quite be reached. Even so, many ionic equilibria can be made so lopsided that, as in a homoeopathic remedy, there is only a small statistical chance of having even a single ion of a particular species in the solution. The Nernst negative potential is correspondingly great.

So Daedalus feels that homoeopathic remedies work, quite accidentally, by evoking Nernst potentials in the patient's tissues. They must contain impurities or trace components which release tiny concentrations of ions into the body. Common ions, such as sodium and calcium, would be swamped by the body's natural levels. Only rare and unusual ions could work: perhaps by acting on electrochemical organs such as nerves. Hence, of course, the popularity of homoeopathic treatment for the more nebulous and nervous afflictions.

DREADCO's electrochemists are now analysing homoeopathic remedies in search of traces of such non-biochemicals as dysprosium, rhenium and various exotic organometallic ions. From the results, they will design a range of rational electrochemical remedies containing traces of ions not usually present in the body, and capable of setting up high Nernst potentials across its cell walls and nerve membranes.

DREADCO's 'homoelectric remedies' should compete strongly with the traditional varieties. Like these, their high dilution will make them very economical in raw materials. But as rational medicines, they will have an intelligible biochemical action. As a result, and in strong contrast to the suspicious innocuousness claimed for so many 'natural' remedies, they may have side effects.

David Jones

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