The model for almost all seasons?

The solution by Onsager, almost exactly half a century ago, of a simple-minded two-dimensional model of a ferromagnet has been a powerful stimulus in theoretical physics.

A MODEL is an approximate description of reality, right? And a good model is a model which, on the one hand, is comprehensible in the sense of providing an image for the mind and which, at the same time, is calculable. Everybody agrees.

The most familiar example is the model of the perfect gas as an assemblage of rigid elastic spheres, billiard balls in textbook language. Once Maxwell had shown that a collection of such spheres would indeed obey the rules required of a perfect gas, the rigid-sphere model of kinetic theory was destined for a virtually infinite life. Much of what has since been called physical chemistry consists of the definition of the ways in which atoms and molecules cannot be accurately represented by rigid elastic spheres, and of the offering of explanations.

In retrospect, it would have been good if Maxwell and his contemporaries had been a little more adventurous. The best part of half a century went by before people started looking seriously at the properties of the rigid-sphere liquid, for example. But for the purposes of the kinetic theory, and once the supposition that atoms are rigid elastic spheres has been shown to be a good first approximation, why not have gone a step or so further, and have tested the model to destruction?

Over the years, that has inevitably been done. Relaxing the condition that the collision between atoms should be elastic, or that momentum should be conserved, leads straightforwardly to predictions of the departure of real billiard-ball gases from the laws of the perfect gas and to the calculation of the virial coefficients in terms of the parameters of inelasticity, whatever they may have meant physically. Similarly, it would have been good to know what the late nineteenth century made of the non-spherical, perhaps ellipsoid, model of the atom. There may still be essays in this direction by disappointed authors still to be rescued from late-Victorian attic rooms.

It is difficult to think of a more robust (in the sense of a more generally applicable) model of the real world unless it is the tautologous continuum model of three-dimensional reality. Formally at least, you can calculate almost anything that way. The deformation of a solid object under the influence of an external field of force? No problem.

Simply imagine that the body is made up of a collection of parallelipipeds cemented together by internal forces, calculate the deformation of each of them in terms of the coefficients of the supposed elasticity tensor, put all of them together and require that the forces across each face cancel out. The outcome is a differential equation of some equivalent thereof. But a little reflection will show that these elegant ways of talking, which brought great fame to late-Victorian Cambridge, are not much more than ways of doing calculus in three dimensions (whence the term "tautologous"). They have little to say about physics.

If the three-dimensional continuum is an arid venue to explore, there is luckily a class of models that compare well, in utility and versatility, with the simple hard elastic sphere as a model of an atom in a perfect gas: the Ising model, as this potential paragon among physical models is known. Ising, for what it is worth, was an Austrian who, in the early 1920s, believed it would serve the public interest if there were a simple and stylized model of a ferromagnet that would be mathematically tractable.

So why not take, to begin with at least, the simplest possible case, that of a three dimensional cubic lattice, and site a little magnet at each vertex? Further simplifications immediately suggest themselves. If the little magnets represent real dipole magnets, with fields of force declining as the third power of the distance, surely it will suffice to count only nearest neighbour interactions. And then, with the arrival of quantum mechanics and the doctrine of electron spin, surely it will make sense to suppose that the little magnets can point in only one direction (that of the external field) or the opposite?

So there emerges the Ising model of a quantised ferromagnet: little dipole magnets at the vertices of a regular lattice, and able to point in one direction or the other. There need not be an external magnetic field, but, on the other hand, there may be. The problem is to calculate the degree of magnetic ordering within the three-dimensional array. It sounds simple. As it happens, the problem is insoluble. For the best part of half a century, people have broken their heads on it, but have failed to solve it. Worse, there seems to be no proof that the conundrum cannot be solved analytically, which means that people keep on breaking their heads.

So how can such a model serve any useful purpose? The simple answer is that the model is at least an easy means of defining problems and of representing them mathematically. You might say, for example, that the energy of interaction between two neighbouring sites in the lattice is either some quantity J or $-J$, depending on whether the little magnets are pointing in opposite or the same directions. To make the ferromagnetic model into that for an anti-ferromagnetic, simply make J negative. And so on. So what a pity that the model is incalculable.

That is how it appeared to have remained until the early 1940s. By then, it had been recognised that the Ising model would also be a splendid model of binary alloys such as β-brass, whose practical value rests on a marked transition from an ordered (and tough) state to one in which the lattice atoms are randomly arranged, and which is a malleable metal by comparison. And if only the model could be solved, would it not be possible to make a model of a liquid in equilibrium with its vapour by letting occupied lattice sites represent real atoms, interacting with their neighbouring sites if they are also occupied, but supposing that empty sites represent mere vacuum?

Only in the early 1940s did the late Lars Onsager provide a half solution, a way of calculating the properties of a two-dimensional lattice. Although this calculation was exact, on the face of things it had little to do with real ferro-magnetism; most conspicuously, it did not uncover the finite change of entropy (signalled by the emission of finite amounts of heat on cooling) of real ferromagnets as they are cooled through their transition or Curie temperatures, but only an abrupt change of the specific heat. But that, of course, is how two-dimensional ferromagnets probably behave.

It is quite remarkable what Onsager's demonstration has since accomplished. Almost trivially, given that the energy of a particular configuration of spins on a two-dimensional lattice is a measure of the number of nearest-neighbour sites which are occupied, the technique becomes a way of dealing with random lattice walks. By supposing that interconnections consist of chemical bonds, it is a way of calculating the configurations of polymer molecules.

Ambitions to generalise from two to several possible spin-configurations at every lattice point have been successfully accommodated, while the arrival of high-speed computers has made possible numerical approximations to the solution of the three-dimensional problem. Add in the remarkable way in which similar calculations are now routinely applied to the solution of problems in Quantum Chromodynamics, and you have the basis for a claim that the simple-minded Ising lattice is the most versatile model of them all.

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