[6] Sax, I. N., *Dangerous Properties of Industrial Materials* (second ed.), 596 (Reinhold, New York, 1955).
[7] Flick, D. F., O'Dell, R. G., and Childs, V. A., *Poultry Sci.*, **44**, 1460 (1965).
[8] Peakall, D. B., *Nature*, **216**, 505 (1967).
[9] Conney, A. H., and Klutch, A., *J. Biol. Chem.*, **238**, 1611 (1963).
[10] Kuntzman, R., Welch, R., and Conney, A. H., *Adv. Enzyme Regulation*, **4**, 149 (1966).
[11] Flax, M. H., and Himes, M. H., *Physiol. Zool.*, **25**, 297 (1952).
[12] Ritter, C., Di Stefano, H. S., and Farah, A., *J. Histol. Cytochem.*, **9**, 97 (1961).
[13] Fouts, J. R., and Rogers, L. A., *J. Pharmacol. Exp. Therap.*, **147**, 112 (1964).
[14] Ortega, P., *Lab. Invest.*, **15**, 657 (1966).

Connectance of Large Dynamic (Cybernetic) Systems: Critical Values for Stability

MANY systems being studied today are dynamic, large and complex: traffic at an airport with 100 planes, slum areas with 10^4 persons or the human brain with 10^{10} neurones. In such systems, stability is of central importance, for instability usually appears as a self-generating catastrophe. Unfortunately, present theoretical knowledge of stability in large systems is meagre: the work described here was intended to add to it.

Most of these large systems, often biological or social, are grossly non-linear, which increases the difficulties associated with them. Here we consider linear systems merely as a first step towards a more general treatment.

We have attempted to answer: What is the chance that a large system will be stable ? If a large system is assembled (connected) at random, or has grown haphazardly, should we expect it to be stable or unstable ? And how does the expectation change as n, the number of variables, tends to infinity ?

Monte Carlo-type evidence[1,2] had suggested that the probability of stability decreased rapidly as n was increased, in some cases perhaps as fast as 2^{-n}, an exponentially-fast vanishing of the chance that the system will be stable. This result, however, was for systems that were fully connected, where every variable had an immediate effect on every other variable. While this case is obviously important in theory, it is not the case in most large systems in real life: not every person in a slum has an immediate effect on every other person, and not every cell in the brain directly affects every other cell. The amount of connectedness ("connectance") is often far below 100 per cent. We have studied how such incomplete connectance affects the probability of a system's stability.

Let the linear system's state be represented by the vector $\mathbf{x}$ $(= <\mathbf{x}_1, \ldots, \mathbf{x}_n>$, where each $\mathbf{x}_i$ is a variable, a function of time), and its changes in time by the matrix equation

$$\mathbf{x} = A\mathbf{x}$$

To "join the variables at random" is to give the elements in A values taken from some specified distribution. "Non-connexion from $\mathbf{x}_i$ to $\mathbf{x}_j$" corresponds to giving the element a_{ji} the value zero. Thus, if the specified distribution has a peak at zero, sampling from it will give the equivalent of a dynamic system with many non-connexions. The connectance, C, of the system can then be conveniently defined as the percentage of non-zero values in the distribution. Thus, if the coefficients are drawn from a distribution with 99 per cent zeros, and if $n = 1{,}000$, then each line of the equation would contain about ten non-zero coefficients, corresponding to a system in which each variable is directly affected by about ten other variables.

Because our work was essentially exploratory, we used the distribution in which the non-zero elements were distributed evenly between $-1\cdot0$ and $+1\cdot0$. The elements in the main diagonal, corresponding to the intrinsic stabilities of the parts, were all negative, distributed evenly between $-1\cdot0$ and $-0\cdot1$. Thus each sampled value of A corresponded to a system of individually stable parts, connected so that each part was affected directly by about C per cent of the other parts.

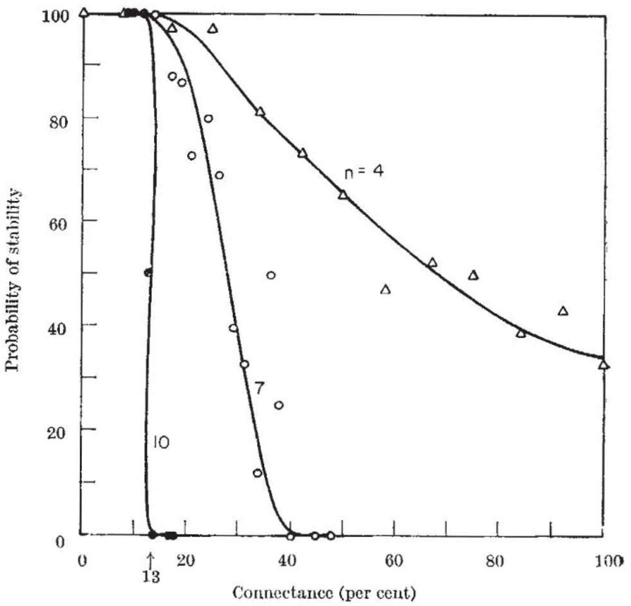

Fig. 1. Variation of stability with connectance.

On a digital computer, a value for n was given and a value for C. Random numbers appropriately distributed were then sampled to provide a matrix A. Hurwitz's criterion was applied to test whether the real parts of A's latent roots were all negative (the stable case) and the result recorded. Further samples, giving further As, allowed the probability of stability (P) to be estimated. The probability was then re-estimated for another value of C, and so on, until the variation of P with C became clear.

The results showed the feature that we wish to report here. As the system was made larger, a new simplicity appeared. Fig. 1 shows a selection of the results, enough to illustrate the principal fact.

When $n = 4$, the probability that the system would be stable depended on C in a somewhat complex curve (which could perhaps be predicted exactly). But as n increases, the curve changes shape rapidly towards a step-function, so that even when n is only 10, the shape might be so regarded, at least for some practical purposes. Thus, even at $n = 10$, questions of stability can be answered simply by asking whether the connectance is above or below 13 per cent: 2 per cent deviation either way being sufficient to convert the answer from "almost certainly stable" to "almost certainly unstable".

The matter is being investigated further, but it may be of general interest to notice that this work suggests that all large complex dynamic systems may be expected to show the property of being stable up to a critical level of connectance, and then, as the connectance increases, to go suddenly unstable.

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MARK R. GARDNER
W. ROSS ASHBY*

Biological Computer Laboratory,
University of Illinois,
Urbana, Illinois 61801.

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* Present address: University College, Cardiff CF1 1XL.

[1] Ashby, W. R., *Design for a Brain* (Chapman and Hall, London, 1952).
[2] Gardner, M. R., *Critical Degenerateness in Linear Systems*, Tech. Rep. No. 5.8 (Biological Computer Laboratory, University of Illinois, Urbana, Illinois 61801, 1968).