LETTERS TO THE EDITOR

PHYSICS

Water Waves and Hamilton's Method

OCEAN waves approaching a beach may be discussed by the method of geometrical optics; the frequency having been assigned, the phase velocity is determined by the depth; the rays are determined by Fermat's principle, and the wave crests are the orthogonal trajectories of the system of rays[1]. But it may not be generally realized that Hamilton's optical method, suitably generalized, can be used to treat a much wider class of problems of water waves. The method is essentially Huygens' construction put into mathematical form.

We start with an equation connecting phase velocity W and wave-length λ:

$$W^2 = \frac{\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda} \left[1 + \frac{T}{g\rho}\left(\frac{2\pi}{\lambda}\right)^2\right]$$

where h is depth, ρ density, and T surface tension. Let y_1, y_2 be wave-numbers so that $(2\pi/\lambda)^2 = y_1^2 + y_2^2$ ($= y^2$, say), and y_3 circular frequency $(2\pi W/\lambda)$; then we have the equation:

$$\Omega \equiv y_3^2 - \tanh hy \, (gy + y^3 T/\rho) = 0 \qquad (1)$$

This we may call the 'Hamilton equation'; regarded as a surface in y-space, it is the analogue of what Hamilton called the "surface of components of normal slowness". Let x_1, x_2 be space co-ordinates, x_3 the time and ψ the phase. Replacing y_r by $\partial\psi/\partial x_r$ in equation (1) ($r = 1, 2, 3$), we get for ψ a partial differential equation which is the analogue of the so-called Hamilton–Jacobi equation. The solution of wave-problems is merely a matter of solving it. This is done by first finding the rays[2] which satisfy the canonical equations $\mathrm{d}x_r/\mathrm{d}w = \partial\Omega/\partial y_r$, $\mathrm{d}y_r/\mathrm{d}w = -\partial\Omega/\partial x_r$, w being a parameter. Then:

$$\psi(x) - \psi(x') = \int_{x'}^{x} y_r \, \mathrm{d}x_r$$

the integral being taken along the ray joining the events x'_r and x_r, with the summation convention. This phase difference is essentially Hamilton's principal function.

As a simple example take $h = \infty$, $T = 0$; the Hamilton equation is then $y_3^2 - gy = 0$, and the rays are $x_1 - x'_1 = -wgy_1/y$, $x_2 - x'_2 = -wgy_2/y$, $x_3 - x'_3 = 2wy_3$, $y_r =$ constant. The phase difference is $\psi(x) - \psi(x') = g(x_3 - x'_3)^2/4\xi$, where $\xi^2 = (x_1 - x'_1)^2 + (x_2 - x'_2)^2$. From this we obtain by simple algebra the familiar Kelvin ship-wave pattern, putting $\psi = 0$ at the prow, for which $x'_1 = Ux'_3$, $x'_2 = 0$, and eliminating x'_r by a condition of stationary phase.

Just as the ordinary method of geometrical optics permits approximate solution of a class of problems far wider than those for which Maxwell's equations can be solved, so the above Hamiltonian method can be used to discuss a very wide range of hydrodynamical problems including those for which h and T vary with position, and even with time. The Hamilton equation can be modified to deal with tidal currents. The most fundamental results in geometrical optics occur in reflexion or refraction at a surface of discontinuity; analogues arise in hydrodynamics at a sharp change in depth, or surface tension (for example, at the edge of an oil-slick), or tidal current. The method deals only with phase; conclusions about amplitude can only be inferred indirectly.

Note added in proof. It may be useful to have a word, analogous to 'photon', 'phonon' and 'graviton', for a fictitious particle which travels with the ray (or group) velocity and carries energy: we suggest the name 'hydron'.

W. F. C. PURSER
J. L. SYNGE

School of Theoretical Physics,
Dublin Institute for Advanced Studies,
64 Merrion Square, Dublin, 2.

[1] Munk, W. H., and Traylor, M. A., *J. Geol.*, **55**, 1 (1947); *Breakers and Surf, Principles in Forecasting*, U.S. Navy Hydrographic Office, Pub. No. 234 (1958).

[2] Synge, J. L., *Handbuch der Physik*, **3**, 1, 124 (1960).

A New Method for the Determination of Plasma Electron Temperature and Density from Thomson Scattering of an Optical Maser Beam

THE Thomson scattering of light by free electrons has not yet been observed in a laboratory plasma—the cross-section for the process is small and until recently no suitable source of light has been available. The advent of a very bright monochromatic light source in the form of the optical maser has markedly changed the situation, however, and in this communication a method is proposed for measuring the temperature and the relative density of electrons in a plasma from the Doppler broadening and the intensity of light scattered from an optical maser beam.

Ruby optical masers already constructed have delivered pulses of 10^4 W. of parallel monochromatic light at 6943 Å. lasting 200 µsec.[1]. Multiple reflexions of such a beam between a mirror placed at the far side of a plasma and the highly reflecting front surface of the optical maser crystal could enhance the beam power by a factor of ten or more, because attenuation by the plasma is slight. Despite the very small cross-section for Thomson scattering ($6{\cdot}65 \times 10^{-25}$ cm.^{2} per electron), with an effective beam-power of 10^5 W. it becomes feasible to detect light scattered from a moderate number of electrons. For example, even at the rather low plasma electron density of 10^{10} cm.$^{-3}$, in 200 µsec. about $4{\cdot}7 \times 10^5$ photons would be scattered from 1-cm. length of beam, and if 1/20 of these were received by a photomultiplier of 4 per cent quantum efficiency a signal-to-noise ratio of more than 30 would be obtained, provided contributions from particles other than electrons may be neglected. The scattering cross-sections at 6943 Å. for most other particles are even smaller, though for neutral atoms of the alkali metals they are comparable.

It will be possible to distinguish between the scattered light and the self-luminosity of the plasma, provided the latter is not overwhelmingly bright, because light scattered through 90° is fully plane-polarized perpendicular to the scattering plane, whereas the self-luminosity of the plasma will generally be unpolarized, or nearly so. Thus the difference in power between components of the light reaching the detector polarized perpendicular to and parallel to the scattering plane will be due to the Thomson scattered light. A polarizing beam splitter and matched photomultiplier channels could con-