PHYSICS

A Very General Class of Exact Solutions in Concentration-dependent Diffusion

THE mathematics of concentration-dependent diffusion has been found in recent years to be important in problems of diffusion in polymers[1–3] and in the problems of the transport of water in its various phases in soils and other porous materials[4–7]. Almost without exception, it has been necessary to use numerical methods to secure the necessary solutions[2,3,8,9]. According to Crank[10], the only known 'formal solutions' are due to Fujita[11,12]. They provide solutions of the equation :

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(D(\theta)\,\frac{\partial\theta}{\partial x}\right) \qquad (1)$$

subject to the conditions :

$$\begin{aligned} \theta = 0,\ \ x > 0,\ \ t = 0\,; \\ \theta = 1,\ \ x = 0,\ \ t > 0\,; \end{aligned} \qquad (2)$$

and are for the following $D(\theta)$ functions :

$$D = D_0/(1 - \lambda\theta);\ D = D_0/(1 - \lambda\theta)^2;$$
$$D = D_0/(1 + 2a\theta + b\theta^2).$$

Here we report the existence of a very general class of $D(\theta)$ functions which yield exact solutions of (1) subject to conditions (2). Fujita's $D(\theta)$'s are contained in this class. A similar very general class of $D(\theta)$ functions yields exact solutions of (1) subject to another set of practically important governing conditions:

$$\theta = 0, x > 0 \text{ and } \theta = 1, x < 0, t = 0;$$
$$\int_0^1 x\,\mathrm{d}\theta = 0, t \geqslant 0. \qquad (3)$$

The substitution:

$$\varphi = xt^{-1/2}$$

enables (1) subject to (2) or (3) to be reduced to[8]:

$$\int_0^\theta \varphi \mathrm{d}\theta = -\,2D\,\mathrm{d}\theta/\mathrm{d}\varphi.$$

That is:

$$D = -\,\tfrac{1}{2}\int_0^\theta \varphi\,\mathrm{d}\theta\,.\,\mathrm{d}\varphi/\mathrm{d}\theta.$$

It follows that the solution of (1) subject to (2) or (3), $\varphi(\theta)$, will exist in exact form, so long as $D(\theta)$ is of the form:

$$D = -\,\tfrac{1}{2}\int_0^\theta F\,\mathrm{d}\theta\,.\,\mathrm{d}F/\mathrm{d}\theta, \qquad (4)$$

where F is an analytical (I use this adjective to describe infinitely differentiable functions of real variables) function of θ which satisfies the following conditions: (a) the integral $\int_0^\theta F\,\mathrm{d}\theta$ exists; (b) $\mathrm{d}F/\mathrm{d}\theta \leq 0$; (c) for (1) subject to (2), $F(1) = 0$; (d) for (1) subject to (3), $\int_0^1 F\,\mathrm{d}\theta = 0$. (Note that D will be expressible in terms of known functions so long as F is integrable in terms of known functions.)

A very large class of F functions satisfies these conditions, so that, accordingly, there exists a very large class of D functions which lead to exact solutions of (1) subject to the appropriate conditions. There is thus an *embarras de richesse* of exact solutions available. A few very simple results are given in Tables 1 and 2.

Table 1. SOME SIMPLE CASES OF EXACT SOLUTIONS OF (1) SUBJECT TO (2)

D	φ	Remarks
$\frac{n}{2}\theta^n\left(1 - \frac{\theta^n}{1+n}\right)$	$1 - \theta^n$	$n > 0$
$\frac{n}{2}\theta^{-n}\left(\frac{\theta^{-n}}{1-n} - 1\right)$	$\theta^{-n} - 1$	$0 < n < 1$
$\frac{1}{2}\sin^2\frac{\pi\theta}{2}$	$\cos\frac{\pi\theta}{2}$	
$\frac{1}{2}e^{2-\theta}(1 - e^{-\theta} - \theta/e)$	$e^{1-\theta} - 1$	

Table 2. SOME SIMPLE CASES OF EXACT SOLUTIONS OF (1) SUBJECT TO (3)

D	φ	Remarks
$\frac{n(n+1)}{2}\theta^n(1 - \theta^n)$	$1 - (n+1)\theta^n$	$n > 0$
$\frac{n}{2(1+n)}[(1 - 2\theta)^{n-1} - (1 - 2\theta)^{2n}]$	$(1 - 2\theta)^n$	$n = 1, 3, 5, 7, ..$
$\frac{n}{2(1-n)}\theta^{-n}(\theta^{-n} - 1)$	$\theta^{-n} - \frac{1}{1-n}$	$0 < n < 1$
$\frac{e^{\theta}}{2}[(e - 1)\theta + 1 - e^{\theta}]$	$e - 1 - e^{\theta}$	

To this point we have developed these ideas under the restriction that F be analytical. However, the existence of D as a continuous function requires merely that F be differentiable once. Further, if a finite number of discontinuities in D is allowable, we may permit F to be not differentiable at a finite number of points in the θ-interval. That is, F may consist of a finite number of different analytical functions of θ, each covering a separate part of the θ-interval. The scope for using analytical methods is therefore even wider than might appear at first glance.

The range of exact solutions is so great that the $D(\theta)$ functions encountered in practical applications should be capable of accurate representation by the form (4). It would therefore seem that foundations have been provided for placing the theory and practice of much of the mathematics of concentration-dependent diffusion on an analytical basis. These matters are treated further in papers in the press in the *Australian Journal of Physics*.

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