Postulates of Probability

THE concept of probability has always comprised two ideas: that of frequency in an ensemble and that of reasonable expectation. Suppose it is said that in tossing an unbiased coin the probabilities of heads and tails are equal. It is implied that the fraction of tosses giving heads will be the same as the fraction giving tails in a large enough number of tosses. It is implied also that in a single trial there is as much reason to expect one result as the other. The choice of one or the other of these ideas as the primary meaning of probability distinguishes the two main schools of thought in the field.

In some examples of probable inference, however, only the idea of reasonable belief is involved. Suppose it is said that one theory of the origin of our solar system is more probable than another. It is meant that one theory is more entitled to belief than the other, not that our solar system originated more often in one way than the other. The opinion which derives probability from frequency in an ensemble must exclude such inference as this from the theory. The distinction which this requires between the cases to be admitted and those to be excluded will often be very tenuous indeed.

On the other hand, the theory based upon reasonable expectation has offered some difficulty because of the vagueness of this concept. Probabilities become calculable in this theory only after the concept has been supplemented by a number of axioms. The necessity of an axiomatic approach has been forcefully argued by Keynes[1], and sets of axioms have been stated with clarity by him and by Jeffreys[2] and others. But detached from the concept of the ensemble these rules have an appearance of artifice which has hindered the acceptance of this view of probability. The purpose of the present note is to point out that more primitive postulates will suffice. It is only necessary to make more use than has been done heretofore of the Boolean algebra of symbolic logic.

The first of these more primitive postulates concerns the probability that both of two events will occur. Let it be assumed only that this probability is determined in some way by the probability of the first event in the given circumstances and the probability of the second event, given these circumstances and the occurrence of the first event. Suppose a bill has been introduced in one house of a bicameral legislature. The probability that it will pass both houses involves first of all the probability that it will pass the first house. It involves further the probability of passage in the second house, passage in the first house being assumed. In more general terms, let **a**, **b** and **c** denote propositions, **a** being given as hypothesis. Let **c·b** denote the joint proposition **c**-and-**b** and **b·a** the joint proposition **b**-and-**a**. Let **b**|**a** and **c·b**|**a** denote the respective probabilities of **b** and **c·b** on the hypothesis **a**. Similarly let **c**|**b·a** denote the probability of **c** on the hypothesis **b·a**. The postulate made above is that **c·b**|**a** is a function of **b**|**a** and **c**|**b·a**.

It can be shown that, with this postulate, Boolean algebra requires that

$$C\,f(\mathbf{c}\cdot\mathbf{b}\,|\,\mathbf{a}) = f(\mathbf{c}\,|\,\mathbf{b}\cdot\mathbf{a})\,f(\mathbf{b}\,|\,\mathbf{a}).$$

where f is an arbitrary function of a single variable and C is an arbitrary constant. Thus far, probability has been identified only as some measure of reasonable belief. If **b**|**a** is one such measure, then $f(\mathbf{b}\,|\,\mathbf{a})$ is another and may just as well be called the probability. It is immaterial whether we use the symbol **b**|**a** or $f(\mathbf{b}\,|\,\mathbf{a})$. For simplicity we write, therefore,

$$C(\mathbf{c}\cdot\mathbf{b}\,|\,\mathbf{a}) = (\mathbf{c}\,|\,\mathbf{b}\cdot\mathbf{a})\,(\mathbf{b}\,|\,\mathbf{a}).$$

Consideration of the case in which **c** is the same proposition as **b** identifies the constant C with the probability of certainty. This is conveniently, though not necessarily, given the value unity.

The second postulate is this: the probability that a proposition is false is determined by the probability that it is true. The symbolic statement is that $\sim\mathbf{b}\,|\,\mathbf{a}$ is a function of **b**|**a**, where $\sim\mathbf{b}$ denotes the proposition not-**b**. Consistency with Boolean algebra now requires that

$$(\sim\mathbf{b}\,|\,\mathbf{a})^m + (\mathbf{b}\,|\,\mathbf{a})^m = 1,$$

where m is an arbitrary number. If **b**|**a** is a measure of reasonable belief, so also is $(\mathbf{b}\,|\,\mathbf{a})^m$. We may therefore call $(\mathbf{b}\,|\,\mathbf{a})^m$ the probability or, making a choice which is different only in form, we may let $m = 1$ and write

$$\sim\mathbf{b}\,|\,\mathbf{a} + \mathbf{b}\,|\,\mathbf{a} = 1.$$

The application of Boolean algebra to the rules already obtained yields others, of which the most important are

$$\mathbf{c}\cdot\mathbf{b}\,|\,\mathbf{a} + \sim\mathbf{c}\cdot\mathbf{b}\,|\,\mathbf{a} = \mathbf{b}\,|\,\mathbf{a} \text{ and } \mathbf{c}\vee\mathbf{b}\,|\,\mathbf{a} + \mathbf{c}\cdot\mathbf{b}\,|\,\mathbf{a} = \mathbf{c}\,|\,\mathbf{a} + \mathbf{b}\,|\,\mathbf{a}$$

where $\mathbf{c}\vee\mathbf{b}$ denotes the disjunctive proposition **c**-or-**b**.

A longer discussion, including the mathematical derivations, is soon to appear in the *American Journal of Physics*.

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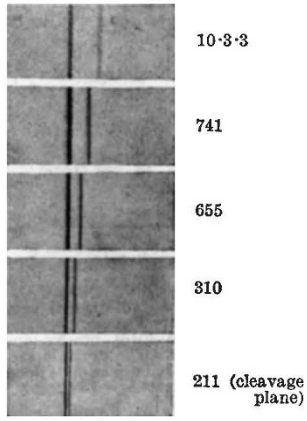
[1] Keynes, J. M., "A Treatise on Probability" (London, 1929).
[2] Jeffreys, H., "Theory of Probability" (Oxford, 1939).

Large Dispersion in X-ray Spectrography Obtained by Using Ground Faces

GROUND faces of calcite can be used with the view of obtaining large dispersion in X-ray spectrography. In a paper published in 1944[1], I showed that a face ground parallel to the lattice plane 655 (Miller's notation in the primitive lattice) yielded good results. The spacing for this plane is 1·032 A. Since then, I have made attempts with faces ground parallel to the planes 741 and 10·3·3, for which the spacings are 0·716$_5$ and 0·533 A. respectively. In the accompanying



Molybdenum Ka_1a_2 in the first order.

reproduction the molybdenum Ka-doublet in the first order is recorded for a series of natural (211 and 310) and ground (655, 741 and 10·3·3) faces on calcite from Iceland. The photographs were obtained in a vacuum spectrograph of the Siegbahn type with an effective radius of 18 cm., 2-fold magnification.

TABLE 1. $\frac{d\theta}{d\lambda}\cdot 10^{-7}$

pqr	d	MoKα_1 $\lambda = 0{\cdot}708$A.	CuKα_1 $\lambda = 1{\cdot}537$A.	ScKα_1 $\lambda = 3{\cdot}025$A.
10·3·3	0·533 A.	12·55	—	—
741	0·716 A.	8·03	—	—
655	1·032 A.	5·16	7·26	—
310	1·532 A.	3·37	3·80	—
211	3·029 A.	1·66	1·71	1·90

The accompanying table shows the values of $\frac{d\theta^1}{d\lambda}$ for five lattice planes of calcite and for three Ka_1 lines (molybdenum, copper and scandium). From these values the calculation of the displacement of a line on the film for a change in λ of $\triangle\lambda = 1$ X.U. is made by means of the formula

$$D_{pqr} = 2 \times 10^{-11} \times \frac{d\theta}{d\lambda} \times R \text{ cm.},$$

where R is the radius of the spectrograph and θ the reflexion angle.

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[1] *K. Vetenskapsakad. Ark. Kemi*, etc., **18** B, No. 14 (1944).

Accuracy of Atomic Co-ordinates Derived from X-ray Data

PREVIOUS work[1,2] on this topic led to methods and formulæ applicable to three-dimensional summation, and it was obviously desirable to extend this to the more limited (and usual) two- and one-dimensional cases. This has now been done, and in a paper, to appear in full elsewhere, I have obtained general results valid for the more simple types of series.

Finite summation errors are shown, in a diatomic case, to be of the same order whatever the dimensions of the series, provided that the resolution is the same in each case; resolution being here defined as the distance between atomic peaks *in the particular synthesis*.

On the basis of the above treatment, it is shown that if a set of observed $|F|$ values is terminated, by real thermal motion, at reciprocal spacing ρ $(= 2\sin\theta)$; the errors in atomic co-ordinates, δ_n, are given by:

$$\delta_n \propto 1/\rho^{(3+n)/2},$$

where n is the number of dimensions of summation. This relation makes possible a comparison of the errors in different structures containing the same type and number of atoms.

The 'artificial temperature factor' method[3,4] of ensuring convergence is also examined, and is shown to introduce errors greater than those for the elimination of which it is applied; suggestions are made, however, as to the way in which corrections may be made for these.

Finally, the effect of experimental errors, in all cases, is reduced to the simple formula:

$$\varepsilon_n < 0{\cdot}66\ \Delta e/(N\sqrt{v_n}).$$

Here $\triangle e$ is the probable experimental error in the $|F|$ values (*c.* 0·6[1]); N is the atomic number of the atom under consideration; v_n is the 'volume' of the repeat unit for dimensions of summation n; ε_n is the mean error in A. of the atomic parameters. In a particular case