some of the most beautiful formations of all, including some fine erratics four or five feet long, and some curious, slender, apparently windswept stalagmites.

A number of true cave pearls have also been found here, together with a large amount of so-called coral formation, making the cave unrivalled in Mendip both from the point of view of size and beauty.

Owing to its size, and the narrowness and intricacy of its upper passages, the survey and photography of the cave have not been easy. Our work has, in addition, been held up by the salvaging excavation undertaken by members of the Society on the site of our museum, the valuable contents of which were destroyed by fire during an enemy air raid on Bristol.

However, the survey has been completed, and we are now concentrating upon obtaining a comprehensive photographic record of the cave, which offers unlimited possibilities in this direction, and upon an attempt to follow the stream still farther into the heart of Mendip past where we now lose it.

ASPECTS OF MATHEMATICAL LOGIC

BY DR. HAROLD JEFFREYS, F.R.S.

IT is recorded that when a pupil asked Confucius what he would do first if he had absolute power, the Master replied "I should reform language". (The development of the theme in the text of the "Analecta" is scarcely worthy of it, but incorporations are suspected.) The history of mathematical logic since "Principia Mathematica" affords an admirable illustration. Even before that great work, the need for unambiguous definitions and for the explicit statement of even the most harmless hypotheses was a main source of inspiration; but later investigators have found that ambiguities remained. In particular, there was a confusion between a symbol and the thing designated by it, and a propositional function was sometimes a property and sometimes what Prof. Willard Van Orman Quine in his recent book, "Mathematical Logic"*, calls a "statement matrix", that is, an expression that would become a statement if it contained names in place of variables. It was hoped also, especially in Russell's popular works, that the actual existence of numbers could be demonstrated in terms of the theory of classes.

It seems to me that such an approach was bound to be unsatisfactory if the scientific use of mathematics was to be justified. For equality of number between classes has to be defined in terms of an empirical method of comparison, and an empirical hypothesis is used in the statement that two classes defined in terms of some property, found similar in one test, will be found similar in another. This hypothesis is so elementary that it has usually passed unnoticed, but if mathematics is justified only for classes satisfying certain axioms, it follows (1) that we cannot significantly speak of the number of individuals with a certain property if the number is liable to change, (2) if there are in the world no classes at all that satisfy the axioms, the whole system breaks down. The fundamental objection to this approach, from the point of view of an empirical scientist, is that we must be able to query and test any empirical statement whatever, and this cannot be done if some such statements are selected and made part of the method of analysis itself.

* Mathematical Logic. By Prof. Willard Van Orman Quine. Pp. xiii+348. (New York: W. W. Norton and Co. Inc.; London: George Allen and Unwin, Ltd., 1940.) 21s. net.

Later writers have mostly abandoned Russell's attempt; the best known is probably Carnap. Axioms are now regarded as abstract statements, and a clear distinction is drawn between a thing and its name. Logic reduces to stating the rules of a language and investigating what kind of statements can be made in the language. Actual demonstration of the existence of structures formally similar to those laid down in the abstract rules is left to the empirical sciences. Even where the rules are not satisfied they can still serve as a useful standard of comparison. The chief aim now is to show that the rules themselves do not lead to contradiction; ordinary language, if not supplemented by rules that have been discovered by persons still living, does lead to contradictions—some are sufficiently elementary to be given in "The Week-End Book".

It is easy to show that if two contradictory propositions are demonstrable (in the ordinary sense) in a language, then every proposition in the language is demonstrable. If we have p and $\sim p$, and we consider any other proposition q, then p entails (p or q); but $\sim p$ and (p or q) together entail q; hence p and $\sim p$ entail q. Similarly, of course, they entail $\sim q$. This result in one form or another occurs in all the modern languages of mathematical logic. Now if every proposition capable of being stated in a language could be proved both true and false, the language would be of little scientific use; and this argument shows that a useful language must contain no contradictions at all. But it also follows that if we can find a proposition in the language that cannot be proved in the language, then the language is consistent. It is not easy to find such propositions; to prove that a proposition cannot be proved is a very different matter from merely failing to prove it. Carnap, however, produces one. But it might happen that every proposition could be proved true or false in the language without any being provable to be both. It has been proved, however, by Gödel that any consistent language that includes arithmetic contains a statement that can be neither proved nor disproved. In his present book, Quine gives a proof that such a statement exists in his system even before arithmetic has been constructed. This is towards the end of the book, and the argument is difficult. But as a result of this type of work we have now much stronger reason than we had for asserting the consistency of logic and mathematics, and we also know that they can never be complete: we can never lay down formal rules that will enable us to decide whether any statement expressible in the language is true. Twenty years ago we might have had doubts about consistency but thought that somehow every proposition could be either proved or disproved, possibly both.

Quine has introduced a novel feature in the treatment of the theory of types, which is much simpler than in Russell and Whitehead's analysis. In the latter the famous contradiction about whether the class of all classes that are not members of themselves is a member of itself or not is resolved by including in the logic of classes a rule that the statement that a class is a member of itself is neither true nor false, but simply meaningless. This led to much complication, because, for example, a real number was defined as a class of rational numbers, and therefore no rational number could be a real number; the real numbers that we ordinarily regard as rational fractions belong to a different type. Quine finds that he can manage with a less drastic criterion. He still finds that certain classes need special treatment, but that he can give a formal rule for recognizing them by inspection of their definitions, and that it is not necessary to deny the meaning of such a class; but it cannot be a member of another class. He is thus able to introduce a universal class V consisting of all things that can be members. This would be impossible in the "Principia" analysis, since no class could include members belonging to different types. We can apparently say now, if we want to, that $0{\cdot}5000\ldots$ is the same thing as $\frac{1}{2}$ and not something different in kind.

Quine's criterion for the recognition of anomalous classes might be compared with the epistemological considerations given in a recent paper by Bridgman. I think that closer inspection would show that the process of constructing them could never be carried out because no consistent order could be given for carrying out the steps. Carnap and Quine both exclude epistemological considerations from their analysis, but I think that without them they lose a valuable source of suggestions, and one that the empirical sciences cannot possibly dispense with.

I would have liked to see some reference to the difficulty in formal expression of logic propounded by Lewis Carroll in "What the Tortoise said to Achilles". The point is that if we know p and (p implies q) we can infer q and proceed to assert q by itself; this is an essential principle of inference. But if we try to state it symbolically and use it, we simply build up longer and longer expressions and never reach a stage where we actually say 'q'. We can *see* what the rule means and act on it, but we cannot state it formally. This is recognized in "Principia". But some of the modern systems try to avoid the notion of meaning altogether and to speak only of symbols as actual specimens of printers' ink, giving rules for substitution of one type of expression for another. We can see what this means, and carry out the various cancellings permitted by the rules. But it seems to me that if the notion of meaning is eliminated, Lewis Carroll's difficulty is reinstated, and the process will only build up longer expressions and never enable a theorem to be asserted by itself. I think that Quine's system retains enough of the notion of meaning to permit an answer to it, but it should be made explicit.

I have been particularly interested in these recent developments because I have been trying to do for induction what Carnap and Quine seem to have done (in different ways) for deduction: to construct a self-consistent formal theory that will enable statements of certain types to be expressed, but such that the theory by itself says nothing about the truth or probability respectively of any empirical proposition. "Principia" assumed some empirical laws, and so, I think, do the 'printers' ink' theories. But for Carnap and Quine logic and mathematics are languages and their study is the analysis of those languages. This is analogous to the only satisfactory interpretation that I can find of the use of numbers to express probabilities; that it is the choice of a language to give more compact and less ambiguous expression than ordinary language can. If this is true in probability theory, it must be true of pure mathematics, which deals with the extreme cases of probability. There are advantages as well as disadvantages when workers follow totally different routes, and nevertheless arrive so near the same destination.