Letters to the Editor

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Arbitrary Character of World-Geometry

PROF. E. A. MILNE, in the important paper[1] in which he gives an account of an invariant distribution of particles forming an expanding universe in flat space-time, has stated that the geometry adopted in cosmological theories may be chosen arbitrarily, the expression of the laws of Nature being relative to the geometry assumed. A similar view has also been expressed by myself[2]. The first enunciation of the idea, however, seems to have been due to Poincaré in quite the early days of relativity. It is interesting in this connexion to observe that there is a very simple method of converting the law of motion of a particle expressed in the geometry of Einstein's theory to the corresponding law expressed in any other geometry.

In general relativity the world-line of any particle is a geodesic, a four-dimensional track satisfying the principle

$$\delta \int ds = 0, \qquad (1)$$

where

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}\, dx_\mu\, dx_\nu.$$

The g's are here functions of $x_1 \, .. \, x_4$, which when given fix the geometry of the manifold; the x's, being arbitrary Gaussian co-ordinates, may be assumed to be the space and time measures of some (usually specially defined) observer. Multiplying by a dimensional constant and, top and bottom, by the element $d\sigma$ of any parameter, we can write the geodesic principle as

$$\delta \int m \sqrt{\Sigma\, g_{\mu\nu} \frac{dx_\mu}{d\sigma}\frac{dx_\nu}{d\sigma}} \, . \, d\sigma = 0. \qquad (2)$$

But in this form the equation can be interpreted in *any* geometry. Thus if $d\sigma$ is the interval of any specified fourfold, (2) becomes a principle of stationary action in that fourfold,

$$\delta \int W d\sigma = 0, \qquad (3)$$

where W, the weighting function of $d\sigma$, is, with given g's, a known function of the co-ordinates and direction-cosines of the (now curved) track at each point. Or if in (2) we write for σ the t of flat space-time, we have Hamilton's principle direct,

$$\delta \int L dt = 0,$$

with the Lagrangian L a known function of co-ordinates and components of velocity. From this the motion in ordinary space of the particle is obtainable in the usual way.

The philosophic implications of such a conversion are considerable. The motion of a particle being described generally as a track of stationary action (of a ray of light, zero action), in

$$\delta \int dA = \delta \int \frac{dA}{d\sigma}\, d\sigma = 0$$

the invariant element of action dA may be factorised in arbitrary ways into action gradient $dA/d\sigma$ and interval $d\sigma$. The latter fixes the geometry and the former is the weighting function W in (3). The physicist working on classical lines naturally adopts the simplest geometry, flat space-time, throwing the burden of accounting for non-uniform motion on the weighting function, which describes in effect a 'field of force'. The relativist, going to the other extreme, throws the whole burden on the geometry. But though these extreme ways are the simplest, the burden clearly can be distributed arbitrarily between W and $d\sigma$, these being adjustable co-factors of the more fundamental thing, action. Action itself, comprising them both, transcends the ideas of geometry.

In a paper published some years ago[3], I have shown that the electromagnetic laws also can be expressed by a principle of stationary action,

$$\delta \int dA = \delta \int \frac{dA}{dV}\, dV = 0,$$

where dV is a four dimensional volume element in the field. The electromagnetic field, therefore, like the gravitational, is obtained by a factorisation of action, but now made differently, the co-factors being action *density* and *volume* element. The former of these effectively specifies the field, for in flat space-time

$$\frac{dA}{dV} \equiv \tfrac{1}{2}\{(\mathbf{e}^2 - \mathbf{h}^2)^2 + 4(\mathbf{eh})^2\}^{\frac{1}{2}}.$$

Since dV, like $d\sigma$, can be used to define a type of geometry, the feature of arbitrariness in the geometry assumed applies to both classes of field.

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[1] *Z. Astrophys.*, **6**, Heft 1–2; 1933.
[2] *Proc. Roy. Soc.*, A, **139**, 349; 1933.
[3] *Proc. Roy. Soc.*, A, **120**, 483; 1928.

Maximum Optical Paths

ERRORS that have once appeared in print have a way of turning up in the most unexpected places. As Dr. Karl Darrow's interesting article on quantum mechanics in *Review of Modern Physics*, **6**, 23, January 1934, is sure to be very widely read in Great Britain, it is not inopportune to refer to an old mistake that he repeats. He states that optical paths are routes sometimes of minimum and sometimes of maximum time, and that for this reason it

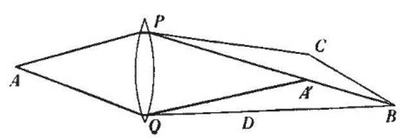

FIG. 1.

is appropriate to refer to them simply as stationary paths. His foundation is wrong though his conclusion is right. The facts are that the time happens to be a minimum when the path does not include an image of an end point of the range considered, but that if the path includes such an image, the time is neither a maximum nor a minimum—it is simply stationary. Thus in Fig. 1, if A', the image of A, is an internal point of the path interval APB, so that the optical