of the casualties is not changed. This is but one of many of the electrical effects produced by light which show the same characteristic. In fact, all these effects indicate that the structure of light must be atomic rather than continuous. If we confine ourselves to the corpuscles, though we might explain the electrical effects we could not explain the optical phenomena of interference; but we must remember that Newton in his confidential moods never contemplates the corpuscle as being the sole constituent of his units. These were always accompanied by vibrations in the ether, and the effect of these as well as the corpuscles must be taken into account.

At the end of the "Optics" come the Queries. In these Newton abandons the severe, almost Euclidean, style of the earlier part of the book; he flings away his policy of "hypotheses non fingo"; he makes up for lost time. The suggestions he makes are extraordinarily acute and suggestive. Here is one of them.

"Are not gross bodies and light convertible into one another and may not Bodies receive much of their activity from the particles of Light which enter into their composition. The changing of bodies into Light and Light into Bodies is very conformable to the course of Nature which seems delighted with transmutations."

In another Query Newton connects the abnormal

refracting powers of some substances with their chemical nature, a subject which is now of great importance. The connexion he suggests is that since these bodies are so affected by light, their chemical nature is such as to make them readily take fire and emit light for themselves. This is the first and most daring of reciprocal relations of the type we are now familiar with in thermodynamics.

Newton suggests that the ether in itself is atomic and that the atoms may not all be of one size. He calculates from the rise of liquid between two glass plates the force exerted by the attraction of the particles of the glass on water at a distance of $\frac{3}{4}$ of one hundred thousandth part of an inch, that is, about one millionth of a centimetre, and finds it sufficient to hold up a cylinder of water two or three furlongs in length. He has extraordinary clear and definite ideas about chemical combination much in advance of anything which appeared for more than a century afterwards.

I have confined myself to Newton's work on optics. I have not time to do more than recall that he was the first to give the theory of the propagation of waves of sound. His work in physics is but a part, and perhaps not the most important part, of his scientific work, but if it stood alone his would still be one of the great names in science.

Newton's Work in Pure Mathematics.¹

By Prof. L. J. MORDELL, F.R.S.

WHEN we consider the numerous and wonderful developments of mathematics since the beginning of the last century, it is very difficult for us to appreciate the state of mathematics just before the rise of Newton. The period was a critical one in the history of this science. All the signs pointed to a great awakening, and the world was ripe for important and far-reaching discoveries. This was the time of the last days of Fermat (1601–1665) and of Descartes (1596–1650), who had both initiated epoch-making discoveries. From the modern point of view, what little was known related to geometry, trigonometry, algebra, and the theory of equations. Their fundamental principles had been laid down roughly in the form in which they are now familiar to elementary students. General theorems were extremely scarce. Each new differential property of a curve, each new expansion of a function of x , required new methods.

¹ Address delivered in King's School, Grantham, on Mar. 19, at the commemoration of the two-hundredth anniversary of the death of Sir Isaac Newton.

Mathematics was chiefly a collection of isolated theorems and examples.

Many distinguished mathematicians have shown unmistakable signs of mathematical genius at an early age. Newton, however, knew no mathematics when he entered Trinity College in 1661 at nineteen years of age. He was introduced to mathematics in his first term by the purchase of a book on astrology, which he could not understand because of the references to geometry and trigonometry. He then started to study Euclid's geometry, which he found very easy and almost obvious. He followed with a book on arithmetic and Descartes's "Géométrie," which was difficult enough to interest him. As an undergraduate he also read the works of Vieta, Van Schooten, and Wallis.

Newton's original investigations were commenced early in his career. In a manuscript of his written in 1665, the year in which he took his B.A. degree, there is the earliest documentary proof of his invention of the fluxional calculus, that is, what is

now called the infinitesimal calculus. He had worked this out fairly completely and had applied it to finding tangents, radii of curvature, etc., of curves. It also led him to the discovery of the binomial theorem.

It is given to very few people, even in their later years when their powers have fully matured, to make so fundamental, so brilliant, so epoch-making a discovery as the invention of the calculus. Thus it is all the more marvellous that Newton should have done so in his early manhood after so short a period of study. His discovery represented the second stage in the emancipation of mathematical ideas from the bondage in which they seem to have been held since the time of the ancient Greeks. The first stage was due to Descartes and his system of analytical geometry.

Newton's discovery meant in those days even more than Cauchy's theory of the complex variable meant in his. It seemed to unlock the gates guarding the storehouse of mathematical treasures; to lay the mathematical world at the feet of Newton and his followers; to open up new paths, new regions that would require years to develop.

It is uncertain how far Newton developed his calculus or what were the dates of his discoveries, as he was very reluctant to publish his results. Many of his most important discoveries had their origin in problems arising from the calculus. Others, especially in geometry, arose from his endeavour to put the proofs of theorems of applied mathematics in a form free from the calculus, which appeared to be greatly distrusted for years after its invention because of the logical difficulties encountered by the mathematicians and philosophers who took up its study.

The question of integration was one of the most fertile sources of Newton's results. Thus in 1676, in letters to Oldenburg, the secretary of the Royal Society, who had been corresponding with Leibniz about some of the latter's results, Newton describes some of his own work and some of his methods of discovery, and in particular of the binomial theorem.

The binomial theorem was discovered in finding the integral of $(1-x^2)^{\frac{1}{2}}$. Newton noted that

$$\begin{aligned}\int (1-x^2)dx &= x - \frac{x^3}{3}, \\ \int (1-x^2)^{\frac{1}{2}}dx &= x - \frac{2x^3}{3} + \frac{x^5}{5}, \\ \int (1-x^2)^{\frac{3}{2}}dx &= x - \frac{3x^3}{3} + \frac{3x^5}{5} - \frac{x^7}{7}.\end{aligned}$$

He studied the forms of the coefficients, and induction suggested to him the result

$$\begin{aligned}\int (1-x^2)^{\frac{1}{2}}dx &= x - \frac{\frac{1}{2}}{3}x^3 + \frac{\frac{1}{2} \cdot \frac{1}{2} - 1}{2!} \frac{x^5}{5} \\ &\quad - \frac{\frac{1}{2} \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} - 2}{3!} \frac{x^7}{7} \dots\end{aligned}$$

He then deduced the binomial expansion for $(1-x^2)^{\frac{1}{2}}$, which he verified by squaring and showing that the result reduced to $1-x^2$. He then gave without proof the expansion of $(a+b)^n$ as an infinite series. It was only after many years that a satisfactory proof was obtained by Abel. Newton realised that infinite series should be convergent, but did not know any tests for convergency. The present-day student would consider it an easy matter to find inductively the well-known expansion of $(1+x)^n$. It comes rather as a shock to realise that the winning of the simple, elementary mathematics of to-day required mighty geniuses at some time or other.

Newton also gave in this letter the expansion of $\sin^{-1} x$. He deduced from this the expansion of $\sin x$ by the method of reversion of series, the earliest example of this process. History was to repeat itself many years afterward when the expansions of elliptic integrals were to precede those of elliptic functions. He also found the length of an arc of an ellipse by infinite series. He evaluated the binomial integral

$$\int x^m(b+cx^n)^p dx,$$

which can be integrated in finite terms if $(m+1)/n$ is a positive integer, and which he thought, erroneously, could not be done otherwise without infinite series. In later years, he also solved the problems of orthogonal trajectories and of the brachistochrone. He solved differential equations by the use of infinite series, and, as he says, any differential equation can be integrated in this way. But this of course does not end the matter.

Very little need be said about the discussion which raged for so many years as to whether Newton or Leibniz deserved priority in the invention of the calculus. It is now considered that they were independent discoverers. This is not really surprising when the work and interests of the mathematicians of their era are considered. The history of mathematics abounds with similar examples. The theory of elliptic functions was discovered independently by Abel and Jacobi, and it was only many years after the death of Gauss that his notebooks revealed that he had anticipated

both of the others, but had not published his work.

Newton lectured on algebra and the theory of equations during the years 1673-1683, and many theorems on these subjects are now associated with his name. The reversion of series has already been mentioned. Another very important theorem, now fundamental in the theory of algebraic functions of a variable, is referred to as Newton's parallelogram. It discusses the shape of a curve with a multiple point at the origin or at infinity and is equivalent to finding from a given relation $F(x, y) = 0$ the initial terms in the expansion

$$y = Ax^{\frac{p}{q}} + Bx^{\frac{r}{s}} + \dots$$

The calculus also led him to the method of interpolation. By this he approximated to the shape of a curve by drawing a higher parabolic curve $y = a + bx + cx^2 + dx^3 + \dots$ through a number of points on the given curve. He was thus enabled to find the area approximately by easy integration.

Newton left his impress particularly upon the theory of equations. He proved that imaginary roots occurred in pairs. It may be recalled that it was only in the nineteenth century that the importance of imaginary numbers was appreciated and fully understood. His formula for the lower limit to the number of imaginary roots in an equation was given without either proof or any indication of the method by which he discovered it. It remained for Sylvester to prove in 1865. Newton laid the foundation for the theory of the symmetrical function of the roots of an equation by giving his well-known formula for the sum of the n th powers of the roots in terms of the coefficients of the equation.

Newton's name has been given to a method for finding approximately the root of an equation. In the form in which it is now used, namely, that if a is an approximation to a root of the equation $f(x) = 0$, then

$$a - \frac{f(a)}{f'(a)}$$

is in general a better approximation, the theorem is not due to Newton but to Raphson.

Newton's method as exemplified by his famous cubic

$$y^3 - 2y - 5 = 0$$

was to put

$$y = 2 + p,$$

so that the equation becomes

$$p^3 + 6p^2 + 10p - 1 = 0.$$

Neglecting the higher powers of p , he gets $10p = 1$. He then puts

$$p = 0.1 + q$$

in the equation for p and gets

$$q^3 + 6.3q^2 + 11.23q + 0.061 = 0.$$

The approximate value of q is given by

$$11.23q + 0.061 = 0.$$

He then puts

$$q = -0.0054 + r,$$

etc.

We come finally to the geometrical work of Newton. He was unequalled by modern writers in his power in the use of the methods of classical geometry, though he discovered no new principles. The geometrical solution of the problem of Pappus, to find the locus of a point such that the rectangle contained by its distances from two given lines shall be in a given ratio to the

rectangle contained by its distances from two other given lines, had baffled the efforts of geometers since the time of Apollonius. Newton apparently gave without much difficulty an elegant proof that the locus was a conic.

Equally skilful was Newton in the application of analytical geometry to the theory of curves. He distinguished between algebraic and transcendental curves, and showed that the former were cut by a straight line in a number of points given by the degree of the equation, and the latter in an infinite number of points. This, of course, is a fundamental theorem in geometry.

Newton investigated the properties of cubic curves analogous to those of conics such as asymptotes and diameters. He then proceeded

FIG. 8.—Cast of Newton's face, in the possession of the Royal Society, London.

From Green's "Short History of the English People."

to reduce the equation of a cubic to four simple canonical forms. Each of these was discussed in detail, and its double points, ovals, etc., were considered. He found seventy-two of the possible seventy-eight forms that a cubic may take. Finally, he stated without proof that all cubic curves could be derived by projection from the canonical form

$$y^2 = ax^3 + bx^2 + cx + d,$$

a theorem which remained unproved until 1731. Apart perhaps from some of his geometrical investigations, Newton's pure mathematics was dominated by his invention of the calculus and his efforts to solve the problems of Nature. No one has surpassed him in combining theory and its application. No one appreciated him more than the greatest of the great mathematicians.

Newton's Work in Optics.

By Sir RICHARD GLAZEBROOK, K.C.B., For. Sec. R.S.

NEWTON entered Trinity College, Cambridge, in 1661 shortly before his nineteenth birthday, and proceeded to his B.A. degree in 1665. On Oct. 1, 1667, he became a fellow of the College, and in 1669 was chosen as Lucasian professor of mathematics in succession to Barrow, whose optical lectures he had helped to edit during the previous year. His own interest in optics began at an early date; while still an undergraduate he had written about haloes. In 1666 he "procured a triangular glass prism to try the celebrated phenomena of colours." His first telescope was constructed in 1668 and his second telescope sent to the Royal Society in 1671. The following year (Jan. 11, 1671, old style) he was elected a fellow of the Royal Society, and his letter containing his new theory of light and colours was read on Feb. 6, 1672. Hooke, who with Boyle and the Bishop of Salisbury was asked to report on the paper, wrote a critical report, claiming that Newton's results were contained in his own "Micrographia" published in 1664. Criticisms were also published by Linus of Liège and Lucas.

Newton was admitted as a fellow of the Royal Society in 1675, and shortly afterwards sent to the Society a paper on the colours of thin plates. Hooke was again a critic. The main part of the work, he said, "was contained in the 'Micrographia,' which Mr. Newton had only carried further in some particulars."

Some correspondence followed, and Newton in his final letter gave Hooke full credit for his work. "You," he wrote, "have added much several ways, and specially in considering the colours of thin plates. If I have seen farther it is by standing on the shoulders of giants."

It was not until some thirty years later (1704) that the first edition of the "Optics" appeared. A second edition with additions—"A Treatise," it is called, "of the Reflexions, Refractions, Inflexions, and Colours of Light"—was printed by the printers to the Royal Society in 1718, fifty years after the construction of his first telescope, and contains, in

a series of Queries of very real interest, his latest views on optics and allied subjects.

But before dealing with the questions raised in these, some reference to the book itself is called for. The difficulties which followed the reading of his papers in 1675 were the main cause of the delay in publication. "To avoid being engaged in Disputes about these Matters," he writes, "I have hitherto delayed the printing, and should still have delayed it had not the Importunity of Friends prevailed upon me." The treatise is divided into three books. The first deals with the reflexion, refraction, and dispersion of light; the second with the colours of thin and thick plates; and the third with diffraction, and in the Queries with double refraction and the theory of light.

The first book opens with a statement of known propositions as a series of definitions and axioms. "For what hath hitherto been generally agreed on I content myself to assume under the Notion of Principles. . . . And this may suffice for an Introduction to Readers of quick Wit and good Understanding not yet versed in Optics, although those who are already acquainted with this Science and have handled Glasses will more readily apprehend what followeth," and then we come to Prop. I. Theor. I. "Lights which differ in Colour differ also in Degrees of Refrangibility." From this we pass by a series of experiments most beautifully designed, most clearly described, to an investigation of the chromatic aberration of lenses with an account of the defects of refracting telescopes, incurable, as Newton thought he had proved it to be by these experiments, and the construction of his own reflecting instrument. "Seeing, therefore," he writes, "that the Improvement of Telescopes of given lengths by Refractions is desperate, I contrived therefore a Perspective by Reflexion, using instead of an Object-glass a concave Metal." The mirrors of the first two telescopes he made and polished himself.

"By many trials I learnt the way of polishing till I made those two reflecting Perspectives. . . .