Greek Geometry, with Special Reference to Infinitesimals.¹

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GREEK geometry passed through several stages from its inception to its highest development in the hands of Archimedes and Apollonius of Perga. The geometry which Thales brought from Egypt early in the sixth century B.C. was little more than a few more or less accurate rules for the mensuration of simple figures; it was the Greeks who first conceived the idea of making geometry a science in and for itself. With Pythagoras and the Pythagoreans, who represent the next stage after Thales, geometry became a subject of liberal education. Apart from special discoveries such as those of the theorem of the square on the hypotenuse, the equality of the three angles of any triangle to two right angles, the construction of the five cosmic figures (the five regular solids), and the incommensurability of the diagonal of a square with its side, the Pythagoreans invented two methods which remained fundamental in Greek geometry, that of proportions (though in a numerical sense only) and that known as application of areas, which is the geometrical equivalent of the solution of a quadratic equation.

By about the middle of the fifth century the Pythagoreans had systematised the portion of the Elements corresponding to Euclid Books I., II., IV., VI., and probably III.

In the second half of the fifth century, concurrently with the further evolution of the Elements, the Greeks attacked three problems in higher geometry, the squaring of the circle, the trisection of any angle, and the duplication of the cube. Hippias of Elis first trisected any angle by means of his curve, the *quadratrix*, afterwards used to square the circle. Hippocrates of Chios, who also wrote the first book of Elements and a treatise on the squaring of certain lunes, reduced the problem of duplicating the cube to that of finding two mean proportionals in continued proportion between two straight lines, the first solution of which was by Archytas, who used a wonderful construction in three dimensions. Democritus, among many other mathematical works, wrote on irrationals; he was also on the track of infinitesimals, and was the first to state the volume of any pyramid and of a cone.

The fourth century saw the body of the Elements completed. Eudoxus discovered the great theory of proportion set forth in Euclid Book V. and the "method of exhaustion" for measuring curvilinear areas and solids. Theætetus contributed to the content of Book X. (on irrationals) and Book XIII. (on the five regular solids). This brings us to Euclid (*fl.* about 300 B.C.).

To the third century B.C. belong Aristarchus of Samos, who anticipated Copernicus; and Archimedes, who, with Apollonius following after twenty years or so, concludes the golden age of Greek geometry.

To come to the history of infinitesimals. The Pythagoreans discovered the incommensurable and maintained the divisibility of mathematical magnitudes *ad infinitum*. The difficulties to which the latter doctrine gave rise were brought out with in-

comparable force by Zeno in his famous Paradoxes and in other like arguments. Zeno's Dilemmas profoundly affected the lines on which mathematical investigations developed. Antiphon the Sophist, in connexion with attempts to square the circle, declared that by inscribing successive regular polygons in a circle, beginning with a triangle or square and continually doubling the number of sides, we shall ultimately arrive at a polygon the perimeter of which will coincide with that of the circle. Warned by Zeno's strictures, mathematicians denied this and substituted the statement that by following the procedure we can draw an inscribed polygon differing in area from the circle by as little as we please. Similarly they would never speak of the infinitely great or infinitely small; they limited themselves to postulating that by continued division of a magnitude we shall ultimately arrive at a magnitude smaller than any assignable magnitude of the same kind, and that by continual multiplication of any magnitude however small we can obtain a magnitude exceeding any assignable magnitude of the same kind however great. On this safe basis Eudoxus founded the whole of his theory of proportion and the method of exhaustion.

It has been remarked that the method of exhaustion, though a conclusive method of proof, requires previous knowledge of the result to be proved, and is of no use for discovering new results. This is scarcely true because, before the proof by *reductio ad absurdum* is applied, the area or volume has to be *exhausted*, and this process often indicates the result. The process means a summation of a series of terms; and there are different classes of cases according to the nature of the series to be summed. In one case (Archimedes' quadrature of a parabolic segment) the summation is that of the geometrical progression $1 + \frac{1}{4} + (\frac{1}{4})^2 + \dots$. Archimedes sums this, nominally, to n terms only. He says nothing about taking the limit when n is increased indefinitely, but merely declares that the area of the segment, which is actually $A\{1 + \frac{1}{4} + (\frac{1}{4})^2 + \dots \text{ad inf.}\}$, is $\frac{4}{3}A$, where A is the area of a certain triangle. It seems plain, nevertheless, that Archimedes found his result by mentally taking the limit. Other series summed by him are arithmetical progressions and the series of the squares of the first n natural numbers. In these cases Archimedes sums two series representing respectively figures circumscribed and inscribed to the figure to be measured, and then states a certain intermediate quantity to be the actual area or content required. Here again Archimedes, though he does not say so, states what is in fact the common limit of the two sums when the number of terms in the series is indefinitely increased, and a factor common to all the terms is at the same time indefinitely diminished. The result is actually equivalent to integration. There are some six cases of the kind depending on the integrations $\int x dx$, $\int x^2 dx$, $\int (ax + x^2) dx$ and $\int \sin \theta d\theta$ taken between proper limits respectively.

The reasons why the Greeks were so limited in the number of integrations which they could directly effect were that they had no algebraical notation and

¹ Abridged from the presidential address to the Mathematical Association, January 2.

had not discovered the modern developments of certain functions as series; nor had they discovered that differentiation and integration are the inverse of one another. There is little trace in Greek geometry of considerations corresponding to the differential calculus; the only case that seems certain is that of the subtangent property of a spiral which must have been obtained by the consideration of the instantaneous direction of motion, at any point on the curve, of the point describing it. If, as is probable, Apollonius, in his treatise on the *cochlias* or cylindrical helix, dealt with tangents to the curve, he would no doubt determine the direction of the tangent at any point in the same way.

But the Greeks were by no means limited to what they could obtain by direct integration. They were very ingenious in reducing an integration which they could not perform directly to another the result of which was already known. This must have been the method by means of which Dionysodorus found the content of an anchor-ring or tore and Pappus obtained his theorem which anticipated what is known as Guldin's theorem. In the matter of the anchor-

ring the Greeks also anticipated Kepler's idea that the content is the same as if the ring be conceived to be *straightened out* and so to become a cylinder. The Method of Archimedes is mostly devoted to the reduction of one integration to another the result of which is known, but is remarkable also as showing how he obtained certain results otherwise proved in his main treatises. The method was a mechanical one of measuring elements of one figure against elements of another, the elements being expressed as parallel straight lines in the case of areas and parallel plane sections in the case of solids. This point of view anticipated Cavalieri. The elements are really infinitesimals, indefinitely narrow strips and indefinitely thin laminae respectively, though Archimedes does not say so. But Archimedes disarms any criticism that could charge him with using infinitesimals for proving propositions by carefully explaining that the mechanical procedure does not constitute a proof and is only useful as *indicating* the results, which must then, before they are definitely accepted, be proved by geometrical methods, that is, by the method of exhaustion.

The Disappearing Gap in the Spectrum.¹

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II.

TURNING to Fig. 1, B, which is repeated here for convenience of reference, this shows the various outposts where from time to time spectral lines have been located. It will be seen that there is still a considerable gap between 16.35, the limit obtained with the vacuum grating at the L series of aluminium, and

so far. If we consider any typical characteristic X-radiation of an element, for example, the K-radiation, it is found to consist of a number of spectral lines which are denoted by the symbols K_α , K_β , K_γ , in order of ascending frequency. In general there are more than three lines, but we shall adopt the symbol K_γ for the line of highest frequency which is observed, and we

FIG. 1.

17.39, the limit with the crystal spectrometer at the L series of zinc. Between these limits no spectral lines are known, but there is evidence of the excitation of such lines, and data have been obtained for the high-frequency limits of spectra in this region.

This evidence depends upon considerations of a somewhat different character from those dealt with

shall denote its frequency by ν_{K_γ} . These lines can be excited in an element by a stream either of high-frequency radiation or of high-velocity electrons reaching it. In either case the lines are not excited separately, but the whole group K_α - K_γ appears simultaneously. It is found that there are simple and important restrictions on the radiation frequencies and on the electron energies which are capable of

¹ Continued from p. 121.